

Semistable reduction for overconvergent F -isocrystals, IV: Local semistable reduction at nonmonomial valuations

Kiran S. Kedlaya

Department of Mathematics, Room 2-165
 Massachusetts Institute of Technology
 77 Massachusetts Avenue
 Cambridge, MA 02139
 kedlaya@mit.edu

July 22, 2010

Abstract

We complete our proof that given an overconvergent F -isocrystal on a variety over a field of positive characteristic, one can pull back along a suitable generically finite cover to obtain an isocrystal which extends, with logarithmic singularities and nilpotent residues, to some complete variety. We also establish an analogue for F -isocrystals overconvergent inside a partial compactification. By previous results, this reduces to solving a local problem in a neighborhood of a valuation of height 1 and residual transcendence degree 0. We do this by studying the variation of some numerical invariants attached to p -adic differential modules, analogous to the irregularity of a complex meromorphic connection. This allows for an induction on the transcendence defect of the valuation, i.e., the discrepancy between the dimension of the variety and the rational rank of the valuation.

Contents

1	Introduction	2
1.1	Local problems	3
1.2	Valuation-theoretic induction	3
1.3	Variation of differential invariants	4
1.4	Structure of the paper	4

2 Preliminaries	5
2.1 Valuations	5
2.2 The Berkovich closed unit disc	7
2.3 Geometry of valuations	13
2.4 Semistable reduction, local and global	15
2.5 Local monodromy representations	17
3 Analysis of p-adic differential modules	24
3.1 Subsidiary radii of convergence	24
3.2 The Robba condition	26
3.3 Frobenius antecedents and descendants	27
3.4 Cyclic vectors	28
3.5 Dwork modules	29
4 Relative analysis	30
4.1 Analytic rings	30
4.2 Adding a second direction	30
4.3 Passage to the generic fibre	31
4.4 Frobenius structures on ∇ -modules	32
4.5 Behavior at the boundary	33
4.6 Variation along a path	34
4.7 More cross-sectional analysis	35
4.8 Dwork modules	40
4.9 Shrinking the domain of definition	42
5 Induction on transcendence defect	44
5.1 Setup: relative geometry	45
5.2 Setup: differential modules	47
5.3 Artin-Schreier characters and Dwork modules	49
5.4 Monodromy representations	53
5.5 Terminal decompositions	57
5.6 Terminal unipotence	60
5.7 Completion of the proof	62
A Appendix	64
A.1 A technical correction	64
A.2 Another technical correction	65
A.3 Removing the discreteness hypothesis	65

1 Introduction

This paper is the fourth and last of a series, preceded by [17, 18, 19]. The goal of the series is to prove a “semistable reduction” theorem for overconvergent F -isocrystals, a class of p -adic

analytic objects associated to schemes of finite type over a field of characteristic $p > 0$. Such a theorem is expected¹ to have consequences for the theory of rigid cohomology, in which overconvergent F -isocrystals play the role of coefficient objects.

In [17], it was shown that the problem of extending an overconvergent isocrystal on a variety X to a log-isocrystal on a larger variety \bar{X} is governed by the triviality of a sort of local monodromy along components of the complement of X . In [18], it was shown that the problem can be localized on the space of valuations on the function field of the given variety, and that it suffices to work in the neighborhood of a valuation which is minimal (height 1, residual transcendence degree 0). In [19], this resulting local problem was solved at monomial valuations (minimal Abhyankar valuations). In this paper, we solve the local problem at nonmonomial minimal valuations; this completes the proof of semistable reduction for overconvergent F -isocrystals, and also yields an analogous theorem for partially overconvergent F -isocrystals.

The introductions of [17, 18, 19] provide context for what is proved in this paper; we will not repeat that context here. Instead, we devote the remainder of this introduction to an overview of the results specific to this paper, and a survey of the structure of the various chapters of the paper.

1.1 Local problems

The problem of (global) semistable reduction is to show that an overconvergent F -isocrystal on a nonproper k -variety can be extended to a log- F -isocrystal on a proper k -variety after pulling back along a suitable generically finite cover. The results of [17, 18] imply that this problem can be localized within the Riemann-Zariski space of valuations of the function field of the variety, and that it suffices to consider neighborhoods of valuations which are minimal (of height 1 and residual transcendence degree 0).

In [19], this problem was solved for monomial (Abhyankar) minimal valuations using a generalization of the p -adic local monodromy theorem to so-called “fake annuli”. However, already for infinitely singular valuations on surfaces this approach is unsuitable, because of the lack of convenient lifts from characteristic p to characteristic 0 in neighborhoods of such valuations.

1.2 Valuation-theoretic induction

Our strategy for proving local semistable reduction at nonmonomial minimal valuations is to proceed by induction on the transcendence defect (the discrepancy in Abhyankar’s inequality) of the valuation, using the monomial case as a base. Namely, given an irreducible variety X and a minimal nonmonomial valuation v on the function field $k(X)$, after altering (in the sense of de Jong) we can construct a dominant morphism $X \rightarrow X^0$ of relative dimension 1, such that the restriction v^0 of v to $k(X^0)$ has the same rational rank as v . Consequently,

¹This expectation has been confirmed by Caro and Tsuzuki [6], who use the semistable reduction theorem to show that Caro’s category of overholonomic arithmetic \mathcal{D}^\dagger -modules both contains the overconvergent F -isocrystals and is stable under standard cohomological operations. See also [5].

the transcendence defect of v^0 is one less than that of v . Moreover, we can construct a path in valuation space terminating at v and otherwise passing only through valuations of lower transcendence defect. Using this path, we can reduce local semistable reduction at v to local semistable reduction at a valuation of lower transcendence defect; this uses an analysis of variation of differential invariants, described below.

Before continuing, we should add some remarks about this valuation-theoretic induction argument. While it seems quite natural, we did not find any analogous argument in the literature before preparing this paper. However, after this paper was largely completed, we learned of Temkin’s proof of local uniformization for any function field in positive characteristic after a purely inseparable extension [34], in which a very similar inductive argument is used. We subsequently used the inductive method to describe the formal structure of flat meromorphic connections on complex algebraic and analytic varieties [21]. We suspect that there are additional problems susceptible to this strategy, e.g., in the valuation-theoretic study of plurisubharmonic singularities [4].

1.3 Variation of differential invariants

Assuming local semistable reduction for all valuations of transcendence defect less than that of v , we can then establish local semistable reduction at v by analyzing the variation of a certain quantity associated to the derivation in the fibral direction. This is similar to the analysis of the differential Swan conductor introduced in [15], except that there one considers all derivations. Measuring only in one direction creates some initial difficulties, but ultimately allows to make the analysis at a deep enough level to avoid getting entangled in valuation-theoretic complications.

To simplify the analysis, we split the discussion into two parts. We first consider a highly abstracted situation, using rings with one power series variable over a field admitting a family of distinct norms. Here we obtain a partial analogue of the usual p -adic local monodromy theorem, using detailed results from [20] concerning ordinary p -adic differential equations. We then introduce the additional geometry, which allows us to obtain a closer analogue of the monodromy theorem, and to carry out the induction on transcendence defect.

1.4 Structure of the paper

We conclude this introduction with a summary of the structure of the paper.

In Section 2, we install some preliminary definitions in valuation theory, give a brief discussion of the Berkovich unit disc, state the local and global semistable reduction theorems, and formulate a notion of the (semisimplified) local monodromy representation associated to an F -isocrystal.

In Section 3, we recall a number of facts from the theory of differential modules on a one-dimensional disc or annulus. These are mostly taken from the book [20], which is based on a course given by the author in the fall of 2007.

In Section 4, we make the abstracted relative analysis described above, concerning differential modules over a ring with one power series variable and a coefficient field equipped

with a family of norms.

In Section 5, we specialize the relative analysis to the case coming from an F -isocrystal, and use it to prove local semistable reduction by induction on transcendence defect.

In the Appendix, we record some corrections to [17] and [18], and reflect upon the necessity of the discreteness hypothesis on the coefficient field.

Notation 1.4.1. We retain the basic notations of [17, 18, 19]. In particular, k will always denote a field of characteristic $p > 0$, K will denote a complete discretely valued field of characteristic zero with residue field k , equipped with an endomorphism σ_K lifting the q -power Frobenius for some power q of p , \mathfrak{o}_K will denote the ring of integers of K , and \mathfrak{m}_K will denote the maximal ideal of K .

Acknowledgments

Thanks to Atsushi Shiho for finding and helping to correct the error in [17] reported in the appendix. Thanks also to the referee for detailed comments, particularly on Proposition 4.8.2 and on § 5 as a whole, and for pointing out the error in [18] reported in the appendix. The author was supported by NSF CAREER grant DMS-0545904, a Sloan Research Fellowship, and the NEC Research Support Fund.

2 Preliminaries

In this section, we introduce some valuation theory, then recall the local approach to proving semistable reduction. This culminates with the statement of the global semistable reduction theorem we will be proving.

2.1 Valuations

In order to introduce local analogues of the problem of semistable reduction, we recall some terminology concerning valuations.

Hypothesis 2.1.1. Throughout this subsection, let F be a finitely generated field extension of k , and let $v : F^* \rightarrow \Gamma$ be a surjective Krull valuation over k (i.e., with $k^* \subseteq \ker(v)$). Let \mathfrak{o}_v and κ_v denote the valuation ring and residue field of v .

These definitions are standard; they were first recalled in this series in [18, §2].

Definition 2.1.2. The *Riemann-Zariski space* $S_{F/k}$ is the set consisting of the equivalence classes of Krull valuations on F over k . We topologize $S_{F/k}$ with the *patch topology* (or *Zariski-Hausdorff topology*), in which bases are given by sets of the form

$$\{w \in S_{F/k} : w(x_1) \geq 0, \dots, w(x_m) \geq 0; \quad w(y_1) > 0, \dots, w(y_n) > 0\}$$

for $x_1, \dots, x_m, y_1, \dots, y_n \in F$. If E/F is an arbitrary field extension, then the restriction map $S_{E/k} \rightarrow S_{F/k}$ is continuous and surjective.

Definition 2.1.3. The *height* of v , denoted $\text{height}(v)$, is the rank of the totally ordered group Γ , i.e., the number of isolated subgroups of Γ . The height is 0 only for v trivial; for v nontrivial, the height is 1 if and only if Γ is isomorphic as an ordered group to a subgroup of \mathbb{R} . We will regularly make such an identification implicitly, even though it is only well-defined up to a rescaling. The *rational rank* of v , denoted $\text{ratrank}(v)$, is the rational rank of Γ , i.e., the dimension of $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ as a vector space over \mathbb{Q} . This is always greater than or equal to the height of v .

The following is a consequence of Abhyankar's inequality [18, Theorem 2.5.2], or an older result of Zariski [37, Appendix 2, Corollary, p. 334].

Proposition 2.1.4. *We have*

$$\text{ratrank}(v) + \text{trdeg}(\kappa_v/k) \leq \text{trdeg}(F/k),$$

where $\text{trdeg}(A/B)$ denote the transcendence degree of the field extension A of B . In particular, both quantities on the left are finite. Moreover, if equality holds, then the group Γ is finitely generated.

This definition was introduced in [18, Definition 4.3.2].

Definition 2.1.5. We say v is *minimal* if $\text{height}(v) = 1$ and $\text{trdeg}(\kappa_v/k) = 0$.

This definition is not quite standard; see Remark 2.1.7 below.

Definition 2.1.6. The *transcendence defect* of v is the difference

$$\text{trdefect}(v) = \text{trdeg}(F/k) - \text{ratrank}(v) - \text{trdeg}(\kappa_v/k),$$

which is nonnegative by Abhyankar's inequality. We say v is an *Abhyankar valuation* if $\text{trdefect}(v) = 0$; if v is both minimal and Abhyankar, we say that v is *monomial*.

Remark 2.1.7. While the notion of transcendence defect is quite natural given Abhyankar's inequality, it does not seem to have a generally accepted name. For instance, we used the term *corank* in early drafts of this paper, while the term *defect rank* was used in early drafts of Temkin's paper on inseparable local uniformization [34]. Our present terminology is based on the fact that those valuations for which Abhyankar's inequality becomes an equality, while commonly known as *Abhyankar valuations* in the context of algebraic geometry, are often referred to as *valuations without transcendence defect* in the context of the model theory of valued fields. For instance, this terminology occurs in Kuhlmann's generalization of the Grauert-Remmert stability theorem [26]. The reference to defect also makes a link with other numerical quantities that measure pathological behavior of extensions of valued fields, such as the classical *henselian defect* of Ostrowski, and the *vector space defect* of Green, Matignon, and Pop [9].

For proofs of the following remarks, see [30, Chapter 6].

Remark 2.1.8. For any extension F' of F , there exists at least one extension v' of v to a Krull valuation on F' [18, Definition 2.2.8]. In case F' is an algebraic extension of F , so that $\text{trdeg}(F'/k) = \text{trdeg}(F/k)$, the extension preserves numerical invariants in the following sense. Since $\Gamma_{v'}$ is contained in the divisible closure of Γ_v , we have $\text{height}(v') = \text{height}(v)$ and $\text{ratrank}(v') = \text{ratrank}(v)$. Since $\kappa_{v'}$ is algebraic over κ_v , we have $\text{trdeg}(\kappa_{v'}/k) = \text{trdeg}(\kappa_v/k)$. Putting these together, we deduce $\text{trdefect}(v') = \text{trdefect}(v)$.

2.2 The Berkovich closed unit disc

Since our strategy for proving local semistable reduction will be induction on transcendence defect, we will need a method for comparing a valuation on a field with the restriction to a subfield, in case the transcendence defects differ by 1. For this, it is most convenient to use Berkovich's notion of the closed unit disc over a complete (but not necessarily algebraically closed) nonarchimedean field.

Hypothesis 2.2.1. Throughout this subsection, let F be a field complete for a nonarchimedean norm $|\cdot|_F$, corresponding to a real valuation v_F . Let \mathbb{C} denote the completed algebraic closure of F . Let $\text{Aut}(\mathbb{C}/F)$ denote the group of *continuous* automorphisms of \mathbb{C} over F .

Definition 2.2.2. The Berkovich closed unit disc $\mathbb{D} = \mathbb{D}_F$ consists of the multiplicative seminorms $|\cdot|$ on $F[x]$ which are compatible with the given norm on F and bounded above by the 1-Gauss norm. In particular, the 1-Gauss norm itself is a point $\alpha_{\mathbb{D}}$ of \mathbb{D} , called the *Gauss point*. (We will often use Greek letters to refer to points of \mathbb{D} . When we want to emphasize the fact that $\alpha \in \mathbb{D}$ represents a function $F[x] \rightarrow \mathbb{R}$, we notate it as $|\cdot|_\alpha$.) The *weak topology* (or *Gel'fand topology*) on \mathbb{D} is the weakest topology under which evaluation at any element of $F[x]$ gives a continuous function $\mathbb{D} \rightarrow [0, +\infty)$.

Lemma 2.2.3. For any complete extension F' of F , the restriction map $\mathbb{D}_{F'} \rightarrow \mathbb{D}_F$ is surjective. Moreover, $\mathbb{D}_{\mathbb{C}} \rightarrow \mathbb{D}_F$ induces a bijection $\mathbb{D}_{\mathbb{C}}/\text{Aut}(\mathbb{C}/F) \rightarrow \mathbb{D}_F$.

Proof. See [2, Corollary 1.3.6]. □

Definition 2.2.4. For $z \in \mathfrak{o}_{\mathbb{C}}$ and $r \in [0, 1]$, the function $|\cdot|_{z,r} : F[x] \rightarrow [0, +\infty)$ given by taking $P(x)$ to the r -Gauss norm of $P(x+z)$ is a seminorm; let $\alpha_{z,r}$ be the associated point of \mathbb{D} . In particular, the map $z \mapsto \alpha_{z,0}$ induces an injection $\mathfrak{o}_{\mathbb{C}}/\text{Aut}(\mathbb{C}/F) \hookrightarrow \mathbb{D}$. Define the closed disc $D_{z,r} = \{z' \in \mathbb{C} : |z' - z| \leq r\}$.

Lemma 2.2.5. Let $P \in F[x]$ be irreducible, and suppose $z_1 \in \mathfrak{o}_{\mathbb{C}}$ is a root of P . Suppose $r_1 \in [0, 1]$ and $z_2 \in \mathfrak{o}_{\mathbb{C}}$ are such that $|z - z_2| > r_1$ for each root z of P . Then

$$|P(z_2)| > |P|_{z_1, r_1}.$$

Proof. By applying an element of $\text{Aut}(\mathbb{C}/F)$, we may assume that the minimum of $|z - z_2|$, for z running over roots of P , is achieved by $z = z_1$. By hypothesis, we have $|z_2 - z| > r_1$. On the

other hand, since $|z_2 - z| \geq |z_2 - z_1|$, we have also $|z_2 - z| = \max\{|z_2 - z|, |z_1 - z_2|\} \geq |z_1 - z|$. If we factor $P(x) = c \prod_z (x - z)$, we obtain

$$\begin{aligned} |P(z_2)| &= |c| \prod_z |z_2 - z| \\ &= |c| \prod_z \max\{r_1, |z_2 - z|\} \\ &> |c| \prod_z \max\{r_1, |z_1 - z|\} = |P|_{z_1, r_1}; \end{aligned}$$

here the strict inequality occurs because the inequality $\max\{r_1, |z_2 - z|\} \geq \max\{r_1, |z_1 - z|\}$ is strict for $z = z_1$. \square

Definition 2.2.6. For $\alpha, \beta \in \mathbb{D}$, we say that α *dominates* β , denoted $\alpha \geq \beta$, if for all $P \in F[x]$, we have $|P|_\alpha \geq |P|_\beta$. The relation of dominance is transitive, and the Gauss point $\alpha_{\mathbb{D}}$ is maximal. Dominance is also stable under base change in an appropriate sense; see Lemma 2.2.9 below. (This assertion is made without proof in [2, Remark 4.2.3].) If $F = \mathbb{C}$, then $\alpha_{z,r} \geq \alpha_{z',r'}$ if and only if $D_{z',r'} \subseteq D_{z,r}$: namely, $\alpha_{z,r} \geq \alpha_{z',r'}$ implies $r = |x - z|_{z,r} \geq |x - z|_{z',r'} = \max\{r', |z - z'|\}$.

Here is Berkovich's classification of points of \mathbb{D} ; we will use it repeatedly in what follows without explicit citation.

Proposition 2.2.7. *Each element of \mathbb{D} is of exactly one of the following four types.*

- (i) *A point of the form $\alpha_{z,0}$ for some $z \in \mathfrak{o}_{\mathbb{C}}$.*
- (ii) *A point of the form $\alpha_{z,r}$ for some $z \in \mathfrak{o}_{\mathbb{C}}$ and $r \in (0, 1] \cap |\mathbb{C}^*|$.*
- (iii) *A point of the form $\alpha_{z,r}$ for some $z \in \mathfrak{o}_{\mathbb{C}}$ and $r \in (0, 1] \setminus |\mathbb{C}^*|$.*
- (iv) *The infimum of a sequence α_{z_i, r_i} in which the discs D_{z_i, r_i} form a decreasing sequence with empty intersection and positive limiting radius. (This type does not occur if \mathbb{C} is spherically complete.)*

Moreover, the points which are minimal under domination in $\mathbb{D}_{\mathbb{C}}$ are precisely those of type (i) and (iv).

Proof. For $F = \mathbb{C}$, this is included in [2, 1.4.4] except for the following points.

- We must check that the limiting radius in (iv) must be positive. This holds because \mathbb{C} is complete, so the intersection of any decreasing sequence of closed discs with zero limiting radius is nonempty, and hence defines a point of type (i) and not (iv). In a similar vein, if \mathbb{C} is spherically complete, then any decreasing sequence of closed discs has nonempty intersection, so there are no points of type (iv) at all.

- We must also check that points of type (iv) are minimal. (By contrast, it is obvious that points of type (i) are minimal whereas points of type (ii) and (iii) are not.) Let $\alpha, \beta \in \mathbb{D}$ be such that α is of type (iv) and $\alpha \geq \beta$. Let \mathbb{C}' be a spherically complete algebraically closed extension of \mathbb{C} , which exists by a result of Krull [25, Satz 24]. By Lemma 2.2.3, we can find $\beta' \in \mathbb{D}_{\mathbb{C}'}$ extending β ; since $\mathbb{D}_{\mathbb{C}'}$ has no points of type (iv), we can write $\beta' = \alpha_{z',r'}$ for some $z' \in \mathfrak{o}_{\mathbb{C}'}$ and some $r' \in [0, 1]$.

Let $\alpha_{z_i,r_i} \in \mathbb{D}$ be a decreasing sequence of points with infimum α , such that the D_{z_i,r_i} have empty intersection. Then for any $z \in \mathbb{C}$, we can find i_0 such that $z \notin D_{z_i,r_i}$ for $i \geq i_0$. For such i , we have

$$|x - z|_{z_i,r_i} = \max\{r_i, |z_i - z|\} = |z_i - z|.$$

For $j \geq i \geq i_0$, we have $z_j \in D_{z_i,r_i}$, so $|z_i - z_j| \leq r_i$ and $|z_j - z| = |z_j - z_i + z_i - z| = |z_i - z|$. In particular, $|z_i - z| = |z_{i_0} - z|$, so

$$|x - z|_\alpha = \inf_{i \geq i_0} \{|x - z|_{z_i,r_i}\} = |z_{i_0} - z|.$$

On the other hand, we have

$$\begin{aligned} r_i &= |x - z_i|_{z_i,r_i} \\ &\geq |x - z_i|_\beta = |x - z_i|_{\beta'} \\ &= |x - z_i|_{z',r'} \\ &= \max\{r', |z' - z_i|\}. \end{aligned}$$

Consequently, $|z' - z_i| \leq r_i$. For $i \geq i_0$, we have $|z - z_i| > r_i \geq r'$ and

$$|x - z|_\beta = \max\{r', |z - z_i|\} = |z - z_i| = |x - z|_\alpha.$$

Since \mathbb{C} is algebraically closed, this implies that $|P|_\beta = |P|_\alpha$ for all $P \in \mathbb{C}[x]$, so $\alpha = \beta$.

For general F , note that the action of $\tau \in \text{Aut}(\mathbb{C}/F)$ on \mathbb{C} carries $\alpha_{z,r}$ to $\alpha_{\tau(z),r}$. This implies that the action of $\text{Aut}(\mathbb{C}/F)$ on \mathbb{D} preserves types in the classification. \square

Remark 2.2.8. For $\alpha \in \mathbb{D}$ of type (ii) or (iii), we can always write $\alpha = \alpha_{z,r}$ with $z \in F^{\text{alg}}$. In fact, we can even choose $z \in F^{\text{sep}}$; this follows from the fact that if $P(x) \in F[x]$ is an inseparable polynomial with z as a root, then for $c \in F$ sufficiently small, $P(x) + cx$ is a separable polynomial having a root z' with $|z - z'| < r$.

Lemma 2.2.9. *Suppose $\alpha, \beta \in \mathbb{D}$ satisfy $\alpha \geq \beta$. Then there exist $\alpha', \beta' \in \mathbb{D}_{\mathbb{C}}$ restricting to α, β , respectively, such that $\alpha' \geq \beta'$. (Note that by applying $\text{Aut}(\mathbb{C}/F)$, we may prescribe either α' or β' , but not both.)*

Proof. Suppose first that $\alpha = \alpha_{z_1,r_1}$ and $\beta = \alpha_{z_2,r_2}$ for some $z_1, z_2 \in \mathfrak{o}_{\mathbb{C}}$ and $r_1, r_2 \in [0, 1]$ with $r_1 > 0$. We may choose z_1 to be integral over F , with minimal polynomial P . If there

exist no extensions $\alpha', \beta' \in \mathbb{D}_{\mathbb{C}}$ of α, β satisfying $\alpha' \geq \beta'$, then $|z_2 - z| > r_1$ for each root z of P . By Lemma 2.2.5,

$$|P|_{z_2, r_2} \geq |P|_{z_2, 0} = |P(z_2)| > |P|_{z_1, r_1},$$

contradicting the hypothesis $\alpha \geq \beta$.

Suppose next that $\alpha = \alpha_{z_1, r_1}$ with $r_1 > 0$, but that β is of type (iv). We may repeat the previous argument by constructing a spherically complete and algebraically closed extension \mathbb{C}' of F (as in Proposition 2.2.7), applying Lemma 2.2.3 to construct $\beta' \in \mathbb{D}_{\mathbb{C}'}$ restricting to β , choosing $z_2 \in \mathbb{C}'$ with $\beta' = \alpha_{z_2, r_2}$ for some $r_2 \in [0, 1]$, proceeding as above, then restricting back to \mathbb{C} .

Finally, suppose that α is of type (i) or (iv). Write α as the infimum of a sequence α_{z_i, r_i} with each $z_i \in F^{\text{alg}}$ and each $r_i > 0$; then $\alpha_{z_i, r_i} \geq \beta$ for each i . Let β' be any extension of β to $\mathbb{D}_{\mathbb{C}}$. By the above argument, for each l , we can choose conjugates $z_{1,l}, \dots, z_{l,l}$ of z_1, \dots, z_l such that

$$\alpha'_{z_{1,l}, r_1} \geq \dots \geq \alpha'_{z_{l,l}, r_l} \geq \beta'.$$

Since z_1 has only finitely many conjugates, we can choose a conjugate $z_{1,\infty}$ that occurs as $z_{1,l}$ for infinitely many l . We can then choose a conjugate $z_{2,\infty}$ such that $z_{1,\infty}, z_{2,\infty}$ occur as $z_{1,l}, z_{2,l}$ for infinitely many l , and so on. We thus build a sequence of $z_{i,\infty}$ such that $\alpha'_{z_{i,\infty}, r_i} \geq \alpha'_{z_{i+1,\infty}, r_{i+1}} \geq \beta'$ for all i . The infimum α' of the $\alpha'_{z_{i,\infty}, r_i}$ is then an extension of α dominating β' . \square

Corollary 2.2.10. *If $\alpha, \beta, \gamma \in \mathbb{D}$ satisfy $\alpha > \gamma$ and $\beta > \gamma$, then there exists a single $z \in \mathfrak{o}_{\mathbb{C}}$ and some $r_1, r_2 \in (0, 1]$ for which $\alpha = \alpha_{z, r_1}$ and $\beta = \alpha_{z, r_2}$. In particular, either $\alpha \geq \beta$ or $\beta \geq \alpha$ (so \mathbb{D} forms a tree under domination, rooted at $\alpha_{\mathbb{D}}$).*

Proof. The classification in Proposition 2.2.7 (plus Lemma 2.2.9, which implies that domination can be checked over \mathbb{C}) implies that $\alpha = \alpha_{z_1, r_1}$ and $\beta = \alpha_{z_2, r_2}$ for some $z_1, z_2 \in \mathfrak{o}_{\mathbb{C}}$. If $z_1 \in D_{z_2, r_2}$, then we can replace z_2 by z_1 , and conversely if $z_2 \in D_{z_1, r_1}$. If neither occurs, then $|z_1 - z_2| > \max\{r_1, r_2\}$, so the two discs D_{z_1, r_1} and D_{z_2, r_2} are disjoint. In this case, α and β cannot both dominate γ . \square

Definition 2.2.11. For $\alpha \in \mathbb{D}$, we define the *radius* of α , denoted $r(\alpha)$, as

$$r(\alpha) = \inf\{r \in [0, 1] : \alpha_{z, r} \geq \alpha \text{ for some } z \in \mathbb{C}\}.$$

For $z \in \mathfrak{o}_{\mathbb{C}}$ and $r \in [0, 1]$, evidently $r(\alpha_{z, r}) = r$. For α of type (iv), r is the infimum of the r_i for any sequence α_{z_i, r_i} as in Proposition 2.2.7. Note that if $\alpha \geq \beta$, then $r(\alpha) \geq r(\beta)$.

Lemma 2.2.12. *For any $\alpha \in \mathbb{D}$, for each $r \in [r(\alpha), 1]$, there is a unique point α_r with $r(\alpha) = r$ and $\alpha_r \geq \alpha$. In particular, if $\alpha \geq \beta$ and $r(\alpha) = r(\beta)$, then $\alpha = \beta$.*

Proof. By Lemma 2.2.9, we may reduce to the case $F = \mathbb{C}$. We first prove existence; for this, we need only consider the case $r > r(\alpha)$, as the case $r = r(\alpha)$ is obvious. Let $\alpha_{z_i, r_i} \in \mathbb{D}$ be a decreasing sequence of points with infimum α . Then $\lim_{i \rightarrow \infty} r_i = r(\alpha)$, so we can find i for which $r_i < r$. For such i , we have $\alpha_{z_i, r} \geq \alpha_{z_i, r_i} \geq \alpha$.

We next prove uniqueness. We first note that if $\alpha_r \geq \alpha$ and $\alpha_r \neq \alpha$, then α_r is nonminimal and so by Proposition 2.2.7 cannot be of type (iv). It thus suffices to check that:

- (a) there is at most one α_r of a given radius of type other than (iv) with $\alpha_r \geq \alpha$;
- (b) if $r = r(\alpha)$ and α is of type (iv), then there is no α_r of radius r and type other than (iv) with $\alpha_r \geq \alpha$.

We first prove (a). Suppose $z, z' \in \mathfrak{o}_{\mathbb{C}}$ satisfy $\alpha_{z,r} \geq \alpha$ and $\alpha_{z',r} \geq \alpha$. If α is not of type (iv), then $\alpha = \alpha_{z'',r''}$ for some z'', r'' , so the intersection $D_{z,r} \cap D_{z',r}$ contains z'' . In particular, this intersection is nonempty, so $D_{z,r} = D_{z',r}$ and $\alpha_{z,r} = \alpha_{z',r}$. If α is of type (iv), let $\alpha_{z_i,r_i} \in \mathbb{D}$ be a decreasing sequence of points with infimum α such that the D_{z_i,r_i} have empty intersection. Choose i_0 large enough so that $z, z' \notin D_{z_i,r_i}$ for $i \geq i_0$. As in the proof of Proposition 2.2.7, we then find that $|z_i - z| = |x - z|_\alpha \leq |x - z|_{z,r} = r$, and similarly $|z_i - z'| \leq r$. Again, $D_{z,r} \cap D_{z',r}$ is nonempty, so $D_{z,r} = D_{z',r}$ and $\alpha_{z,r} = \alpha_{z',r}$.

We next prove (b). Suppose α is of type (iv), $r = r(\alpha)$, and $\alpha_{z,r} \geq \alpha$ for some $z \in \mathfrak{o}_{\mathbb{C}}$. Let $\alpha_{z_i,r_i} \in \mathbb{D}$ be a decreasing sequence of points with infimum α such that the D_{z_i,r_i} have empty intersection. Choose i_0 large enough so that $z \notin D_{z_i,r_i}$ for $i \geq i_0$. As in the proof of Proposition 2.2.7, we have $|z_i - z| = |x - z|_\alpha \leq |x - z|_{z,r} = r \leq r_i$, so $z \in D_{z_i,r_i}$. But then the D_{z_i,r_i} have nonempty intersection, giving a contradiction. \square

Definition 2.2.13. By Lemma 2.2.12, for any $\alpha \in \mathbb{D}$, for each $r \in [r(\alpha), 1]$, there is a unique point α_r with $r(\alpha) = r$ and $\alpha_r \geq \alpha$. We define the *generic path* as the subset of \mathbb{D} consisting of these α_r ; it is homeomorphic to an interval with endpoints $\alpha_{\mathbb{D}}, \alpha$.

Lemma 2.2.14. *Let $\alpha \in \mathbb{D}$ be a point of type (iii) corresponding to a disc containing an F -rational point z . Then $F[x - z, (x - z)^{-1}]$ is dense in $F(x)$ under $|\cdot|_\alpha$.*

Proof. We may write $\alpha = \alpha_{z,r}$ for some $r \in (0, 1] \setminus |\mathbb{C}^*|$. Then the r -Gauss norm of any nonzero element $P = \sum_i P_i (x - z)^i \in F[x]$ is always achieved by exactly one monomial $P_j (x - z)^j$, so the series

$$\sum_{h=0}^{\infty} (P_j(x - z)^j - P)^h (P_j(x - z)^j)^{-h-1}$$

converges in the completion of $F[x - z, (x - z)^{-1}]$ under $|\cdot|_\alpha$. We can thus approximate the reciprocal of P using elements of $F[x - z, (x - z)^{-1}]$, so $F[x - z, (x - z)^{-1}]$ is dense in $F(x)$ under $|\cdot|_\alpha$. \square

Corollary 2.2.15. *With notation as in Lemma 2.2.14, the completion of $F(x)$ under $|\cdot|_\alpha$ may be represented as the ring of formal sums $\sum_{i \in \mathbb{Z}} c_i (x - z)^i$ with $c_i \in F$, such that $|c_i|r^i \rightarrow 0$ as $i \rightarrow \pm\infty$.*

Corollary 2.2.16. *With notation as in Lemma 2.2.14, for any interval $J \subseteq (0, 1]$ containing $r = r(\alpha)$, let R_J be the Fréchet completion of $F(x)$ under $|\cdot|_{z,s}$ for $s \in J$. Then for any open interval J containing r and any nonzero $f \in R_J$, there exist $c_i \in F$, $i \in \mathbb{Z}$, $\epsilon > 0$, and an open subinterval J' of J containing r , such that for all $s \in J'$,*

$$|f - c_i(x - z)^i|_{z,s} \leq (1 - \epsilon)|c_i(x - z)^i|_{z,s}.$$

In particular, the union of R_J over all open intervals J containing r is a field.

Proof. By Corollary 2.2.15, we can write $f = \sum_{i \in \mathbb{Z}} c_i(x - z)^i$ for some $c_i \in F$ such that $|c_i|r^i \rightarrow 0$ as $i \rightarrow \pm\infty$. As in Lemma 2.2.14, there is a unique index i maximizing $|c_i|r^i$. Since the ratio $|f - c_i(x - z)^i|_{z,s}/|c_i(x - z)^i|_{z,s}$ is continuous in s , and it is strictly less than 1 for $s = r$, it is also strictly less than 1 for s in some open neighborhood of r . This proves the claim. \square

Corollary 2.2.17. *With notation as in Lemma 2.2.14 and Corollary 2.2.16, for any open interval J containing r and any finite étale extension S of R_J , there exists an open subinterval $J' \subseteq J$ containing r such that the map*

$$\pi_0(S \otimes_{R_J} R_{J'}) \rightarrow \pi_0(S \otimes_{R_J} R_{[r,r]})$$

(which is necessarily injective) is bijective.

Proof. We first check that any factorization of a monic polynomial $P(T) \in R_J[T]$ into monic coprime factors $Q_1, Q_2 \in R_{[r,r]}[T]$ lifts to a factorization in $R_{J'}[T]$ for some J' . Note that for n a positive integer, $f_1, \dots, f_n \in R_J[T_1, \dots, T_n]$, and $a \in (a_1, \dots, a_n) \in R_{[r,r]}^n$ such that $f_i(a) = 0$ for $i = 1, \dots, n$ and $\det(\partial f_i / \partial T_j(a)) \neq 0$, there exist an open interval $J' \subseteq J$ containing r and some $b \in R_{J'}^n$ such that $f_i(b) = 0$ for $i = 1, \dots, n$. (Namely, we construct b using the standard multivariate Newton-Raphson iteration.) We then obtain the factorization assertion by following the proof of the implication from (d') to (e) in [28, Theorem I.4.2].

To deduce the claim from this, apply the primitive element theorem over $\cup_{J'} R_{J'}$, which is a field by Corollary 2.2.16. For some J' , this gives an isomorphism $S \otimes_{R_J} R_{J'} = R_{J'}[T]/(P(T))$ for some monic separable $P(T) \in R_{J'}[T]$. By applying the previous paragraph to lift all factorizations of $P(T)$ from $R_{[r,r]}(T)$ to $R_{J'}[T]$ for some J' , we get the desired result. \square

Lemma 2.2.18. *Let α be a point of \mathbb{D} of type (ii) or (iii). Let $v = -\log \alpha$ be the corresponding real valuation on $F(x)$.*

(a) *If α is of type (ii), then*

$$\text{trdeg}(\kappa_v/\kappa_{v_F}) = 1, \quad \dim_{\mathbb{Q}}((\Gamma_v/\Gamma_{v_F}) \otimes \mathbb{Q}) = 0.$$

(b) *If α is of type (iii), then*

$$\text{trdeg}(\kappa_v/\kappa_{v_F}) = 0, \quad \dim_{\mathbb{Q}}((\Gamma_v/\Gamma_{v_F}) \otimes \mathbb{Q}) = 1.$$

Proof. Write $\alpha = \alpha_{z,r}$ with $z \in F^{\text{alg}}$. If α is of type (ii), pick $\lambda \in F^{\text{alg}}$ with $|\lambda| = r$, otherwise put $\lambda = 1$. There is no harm in replacing $F, F(x)$ with $F(z, \lambda), F(z, \lambda, x)$, respectively, as by Remark 2.1.8, this only replaces each residue field by a finite extension and each value group by a group containing it with finite index.

Suppose then that $z \in F$. If α is of type (ii), then

$$\kappa_v = \kappa_{v_F}(x/\lambda), \quad \Gamma_v = \Gamma_{v_F}.$$

If α is of type (iii), then by Lemma 2.2.14,

$$\kappa_v = \kappa_{v_F}, \quad \Gamma_v = \Gamma_{v_F} \oplus \mathbb{Z} \log r.$$

This gives the desired assertions. \square

2.3 Geometry of valuations

We next introduce some constructions concerning the interplay between valuations and birational geometry of varieties.

Hypothesis 2.3.1. Throughout this subsection (except in the statement of Theorem 2.3.9), assume that k is algebraically closed, let X be an irreducible variety over k , put $F = k(X)$, and let $v : F^* \rightarrow \Gamma$ be a surjective Krull valuation.

This definition is standard; it was first recalled in this series in [18, §2.4].

Definition 2.3.2. The *center* of v on X is the subset of $x \in X$ for which $\mathcal{O}_{X,x} \subseteq \mathfrak{o}_v$. If the center is nonempty, we say that v is *centered on* X ; in this case, the center is an irreducible closed subset of X , which we will identify with the corresponding reduced closed subscheme. If X is proper, then v is necessarily centered on X . If Z is a closed subvariety of X , we will say that v is *centered on* Z if v is centered on X and the center is contained in Z .

The following definitions are new.

Definition 2.3.3. By a *local alteration* of X around v , we will mean a morphism $f : X_1 \rightarrow X$ which is quasi-projective and generically finite, with X_1 irreducible and some extension of v to $k(X_1)$ centered on X_1 . By contrast, an *alteration* is supposed to be dominant, proper, and generically finite. We say a (local) alteration is *separable* if it induces a separable extension of function fields; this is equivalent to asking it to be generically finite étale. (The separability was built into the definition we used earlier in the series, but is not built into de Jong's original definition.)

Definition 2.3.4. Let X be an irreducible variety, let Z be a closed subvariety of X , and let v be a minimal valuation on $k(X)$ centered on Z . We call the pair (X, Z) an *exposure* of v (or say that v is *exposed* by the pair (X, Z)) if the following conditions hold.

- (a) The center z of v is a smooth point of X lying on exactly r components of Z , for $r = \text{ratrank}(v)$.
- (b) There exists a system of parameters a_1, \dots, a_n for X at z such that $v(a_1), \dots, v(a_r)$ are linearly independent over \mathbb{Q} , and the zero loci of a_1, \dots, a_r at z are the components of Z passing through z .

If $f : X_1 \rightarrow X$ is a local alteration, we say that f is an *exposing alteration* of the pair (X, Z) for v if $(X_1, f^{-1}(Z))$ is an exposure of some extension of v to $k(X_1)$.

The following is an analogue of [19, Proposition 6.1.5] for not necessarily monomial valuations.

Lemma 2.3.5. *Assume that k is algebraically closed. Then with notation as in Definition 2.3.4, there exists a closed subscheme Z' containing Z and a separable exposing alteration of (X, Z') for v .*

This is probably also true without enlarging Z , but that enlargement is harmless for our purposes.

Proof. Choose $h_1, \dots, h_r \in k(X)^*$ such that $v(h_1), \dots, v(h_r)$ are positive and linearly independent over \mathbb{Q} . We enlarge Z so that each h_i , viewed as a rational map $X \dashrightarrow \mathbb{P}_k^1$, restricts to a regular map $X \setminus Z \rightarrow \mathbb{G}_{m,k}$. For the rest of the argument, we will repeat the operation of replacing X by a local alteration around v , Z by its inverse image, and v by some extension to the new function field; we describe this for short as “replacing”.

We first replace to force h_1, \dots, h_r to become regular on X . We then apply de Jong’s alterations theorem (see Theorem 2.3.9 below) to replace so that X becomes smooth and Z becomes a strict normal crossings divisor. Suppose that D_1, \dots, D_s are the components of Z passing through z . Let a_1, \dots, a_n be a system of local parameters of X at z , with a_i vanishing along D_i for $i = 1, \dots, s$. Since the Cartier divisor defined by each h_i in a neighborhood of z is a nonnegative integer linear combination of D_1, \dots, D_s , and $v(h_1), \dots, v(h_r)$ generate a subgroup of $v(k(X)^*)$ of rank r , so do $v(a_1), \dots, v(a_s)$. In particular, we must have $s \geq r$, and some r -element subset of $v(a_1), \dots, v(a_s)$ must be linearly independent over \mathbb{Q} ; we may assume without loss of generality that it is $v(a_1), \dots, v(a_r)$.

Construct the $s \times s$ matrix A given by Lemma 2.3.6 below with $c_i = v(a_i)$, and put

$$b_i = \prod_{j=1}^s a_j^{A_{ij}} \quad (i = 1, \dots, s);$$

the nonnegativity of A^{-1} means that each a_i is a product of nonnegative powers of the b_j . We may then replace once more (by blowing up at z) to produce a new variety with local coordinates $b_1, \dots, b_s, a_{s+1}, \dots, a_n$.

At this point, we observe that $v(b_i) = 0$ for $i = r+1, \dots, s$. Under the inclusion $\mathcal{O}_{X,z} \subseteq \mathfrak{o}_v$ of local rings, the residue fields κ_z and κ_v are identified (since both equal k), so the maximal ideal \mathfrak{m}_v of \mathfrak{o}_v must restrict to the maximal ideal of $\mathcal{O}_{X,z}$. Since $b_i \notin \mathfrak{m}_v$, it follows that b_i does not vanish at z for $i = r+1, \dots, s$. Consequently, the components of the inverse image of Z passing through z are parametrized by b_1, \dots, b_r , so v is exposed. \square

Lemma 2.3.6. *Let $r \leq s$ be positive integers. Let c_1, \dots, c_s be positive real numbers such that c_1, \dots, c_r form a basis for the \mathbb{Q} -span of c_1, \dots, c_s . Then there exists a matrix $A \in \mathrm{GL}_s(\mathbb{Z})$ such that A^{-1} has nonnegative entries, and*

$$\sum_{j=1}^s A_{ij} c_j > 0 \quad (i = 1, \dots, r), \quad \sum_{j=1}^s A_{ij} c_j = 0 \quad (i = r+1, \dots, s).$$

Proof. The general case follows from repeated application of the case $r = s - 1$, which is [36, Theorem 1]. \square

Definition 2.3.7. With notation as in Definition 2.3.4, for $i = 1, 2, \dots$, let $f_i : X_i \rightarrow X$ be an exposing alteration of (X, Z) for v . Let S_i be the set of valuations on $k(X)$ admitting extensions centered on $f_i^{-1}(Z)$. We say that the f_i form an *exposing sequence* for v if the S_i form a neighborhood basis of v in the patch topology on $S_{k(X)/k}$. (It is convenient in some cases to allow an exposing sequence indexed by an arbitrary countable set.)

Lemma 2.3.8. *Let $f : Y \rightarrow X$ be an alteration with Y irreducible. Let v be a valuation centered on X , which admits an extension w to $k(Y)$ which is unramified over $k(X)$. Then there exists a commutative diagram*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow f' & & \downarrow f \\ X' & \longrightarrow & X \end{array}$$

in which f' is an alteration and the horizontal arrows are blowups, such that w is centered on Y' at a point at which f' is étale. (In particular, the property of admitting an unramified extension to $k(Y)$ is an open condition in $S_{k(X)/k}$.)

Proof. By the primitive element theorem, we have $\mathfrak{o}_w = \mathfrak{o}_v[t]/(P(t))$ for some monic polynomial $P(t) = t^d + \sum_{i=0}^{d-1} P_i t^i \in \mathfrak{o}_v[t]$ whose discriminant Δ is a unit in \mathfrak{o}_v . Choose X' so that v is centered on X' at a point z' for which $\Delta, \Delta^{-1}, P_0, \dots, P_{d-1} \in \mathcal{O}_{X', z'}$, then take Y' to be the irreducible component of $X' \times_X Y$ which dominates X' . This has the desired effect. \square

Since we have used de Jong's alterations theorem once already, and will use it several times more, we formally recall its statement.

Theorem 2.3.9 (de Jong). *For k an arbitrary field, let X be a reduced separated scheme of finite type over k , and let Z be a reduced closed subscheme. Then there exists a projective alteration $f : X_1 \rightarrow X$ such that X_1 is regular and the reduced subscheme of $f^{-1}(Z)$ is a strict normal crossings divisor. Moreover, if k is perfect, we may force f to be separable.*

Proof. See [8, Theorem 4.1]. \square

It appears difficult to ensure that the alteration f in Theorem 2.3.9 is generically Galois. We are thus forced to consider the following construction.

Definition 2.3.10. Let X, X_1 be irreducible varieties over a field k , and let $f : X_1 \rightarrow X$ be a separable alteration of generic degree d . Let F be the Galois closure of $k(X_1)$ over $k(X)$, and put $G = \text{Gal}(F/k(X))$. Let ι_1, \dots, ι_d be the distinct embeddings of $k(X_1)$ into F ; these carry a transitive action of G , with which we identify G with a subgroup of the symmetric group S_d .

Let Y be the d -fold fibre product of X_1 over X , equipped with the natural action of S_d . The embeddings ι_1, \dots, ι_d define a geometric point η of the d -fold fibre product of X_1 over X , lying over the generic point of X . Note that the stabilizer of η in S_d is exactly G . Let X_2 be the Zariski closure of η in Y , so that X_2 also carries an action of G . Then $g : X_2 \rightarrow X$ is a separable alteration factoring through X_1 , which we call the *Galois closure* of f .

2.4 Semistable reduction, local and global

We now state the local and global semistable reduction theorems, which constitute the principal results of this paper, and explain how they may be derived from previous results

in this series plus the other results of this paper. We start with the local version. (Beware that some definitions from [18] are not correctly formulated in case k is not perfect; see the appendix for relevant corrections, which we also incorporate here.)

Definition 2.4.1. Let $Z \hookrightarrow X$ be a closed immersion of k -varieties, with X irreducible and $X \setminus Z$ smooth. Let \mathcal{E} be an F -isocrystal on $X \setminus Z$ overconvergent along Z . Let v be a valuation on $k(X)$ centered on Z . We say that \mathcal{E} admits *local semistable reduction* at v if after replacing k with $k^{q^{-n}}$ for some nonnegative integer n , there exists an alteration $f : X_1 \rightarrow X$ with X_1 irreducible (which we require to be separable in case k is perfect), and an open subscheme U of X_1 on which every extension w of v to $k(X_1)$ is centered, such that for $Z_1 = f^{-1}(Z)$, $(U, Z_1 \cap U)$ is a *smooth pair* (i.e., U is smooth over k and $Z_1 \cap U$ is a strict normal crossings divisor) and $f^*\mathcal{E}$ extends to a convergent log- F -isocrystal with nilpotent residues on $(U, Z_1 \cap U)$.

Before proceeding, it will be helpful to insert a remark omitted from [18], although it is partly implicit in the proof of [18, Lemma 4.3.1].

Remark 2.4.2. With X, Z, v, \mathcal{E} as in Definition 2.4.1, suppose instead that $f : U \rightarrow X$ is a local alteration with U irreducible, such that $(U, f^{-1}(Z))$ is a smooth pair, $f^*\mathcal{E}$ extends to a convergent log-isocrystal with nilpotent residues on $(U, f^{-1}(Z))$, and at least one extension of v to $k(U)$ is centered on U . Then \mathcal{E} again admits local semistable reduction, by the following argument. From the form of the definition of local semistable reduction, we may assume k is perfect. By Theorem 2.3.9, we can pick a separable alteration $g : X_2 \rightarrow X$ such that $k(X_2)$ contains the maximal separable subextension of the normal closure of $k(U)$ over $k(X)$, and (X_2, Z_2) is a smooth pair for $Z_2 = g^{-1}(Z)$. Then for any extension w of v to $k(X_2)$ and any component of Z_2 passing through the center of w on X_2 , $g^*\mathcal{E}$ is forced to be unipotent along this component. (In case f was not separable, the Frobenius structure on \mathcal{E} can be used to eliminate the inseparable extension of function fields.) By [17, Theorem 6.4.5], \mathcal{E} is log-extendable to a neighborhood of the center of w , and the Frobenius structure extends uniquely.

Theorem 2.4.3 (Local semistable reduction). *With notation as in Definition 2.4.1, \mathcal{E} always admits local semistable reduction at v .*

Proof. By [18, Theorem 4.3.4], it suffices to prove this result in case k is algebraically closed and v is minimal. If $\text{trdefect}(v) = 0$, then v is a monomial valuation, so the claim holds by [19, Theorem 6.3.1]. This allows us to begin an induction on $\text{trdefect}(v)$, which we may conclude by showing that if k is algebraically closed and Theorem 2.4.3 holds for all X, Z, \mathcal{E}, v with v minimal of transcendence defect at most n , then Theorem 2.4.3 also holds when v is minimal of transcendence defect $n + 1$. This claim is the content of Theorem 5.0.1. \square

We next deduce the global semistable reduction theorem. The case of this theorem in which X is proper answers a conjecture of Shiho [32, Conjecture 3.1.8] which was introduced in this series as [17, Conjecture 7.1.2].

Theorem 2.4.4 (Global semistable reduction). *Let $Z \hookrightarrow X$ be a closed immersion of k -varieties, with $X \setminus Z$ smooth. Let \mathcal{E} be an F -isocrystal on $X \setminus Z$ overconvergent along Z . Then after replacing k with $k^{q^{-n}}$ for some nonnegative integer n , there exists an alteration $f : X_1 \rightarrow X$ (which we require to be separable in case k is perfect) such that for $Z_1 = f^{-1}(Z)$, (X_1, Z_1) is a smooth pair and $f^*\mathcal{E}$ extends to a convergent log- F -isocrystal with nilpotent residues on (X_1, Z_1) .*

Proof. This follows from Theorem 2.4.3 via [18, Propositions 3.3.4 and 3.4.5]. \square

Remark 2.4.5. Remember that the usual p -adic local monodromy theorem of André [1], Mebkhout [27], and the author [12] applies to arbitrary modules with Frobenius and connection structures over the Robba ring, not just those that arise from overconvergent F -isocrystals on curves. It should similarly be possible to formulate and prove Theorem 2.4.4 for “modules with Frobenius and connection concentrated on Z ”, with the same proof.

2.5 Local monodromy representations

Just as one may use the p -adic local monodromy theorem to construct local monodromy representations for overconvergent F -isocrystals on curves [13, §4], we may use local semistable reduction to construct a local monodromy representation associated to an overconvergent F -isocrystal and a valuation. We must assume local semistable reduction at that valuation, which for external use is harmless in light of Theorem 2.4.3. However, we cannot omit the hypothesis here, because we use the construction at one valuation in the course of proving Theorem 2.4.3 for another valuation (as part of an induction on transcendence defect).

We will start with a bit of elementary group theory. This will be useful in conjunction with the properties of inertia groups (Definition 2.5.6).

Lemma 2.5.1. *Let G be a finite p -group. Let F be an algebraically closed field of characteristic zero. Let $\tau : G \rightarrow \mathrm{GL}(V)$ be a linear representation of G on a finite-dimensional F -vector space V .*

- (a) *If τ is irreducible, then $\dim(V)$ is a power of p .*
- (b) *If $\dim(V) = 1$, then there exists a nonnegative integer h such that $\tau^{\otimes p^h}$ is trivial.*
- (c) *If τ is nontrivial, then either τ or $\tau^\vee \otimes \tau$ contains a nontrivial subrepresentation of dimension 1.*

Proof. (a) The dimension of any irreducible representation of any finite group on a finite-dimensional vector space over an algebraically closed field divides the order of the group. Hence $\dim(V)$ divides a power of p , and so must be a power of p itself.
(b) The action of G via τ factors through an abelian quotient of G , necessarily of p -power order.

- (c) Since τ is nontrivial, it is nonzero and admits a nontrivial irreducible subrepresentation ψ . By (a), the dimension of ψ equals p^m for some nonnegative integer m . If $m = 0$, then τ contains a nontrivial subrepresentation of dimension 1, and we are done. Otherwise, we may split $\psi^\vee \otimes \psi$ into its trace component and its trace-zero component. The latter has dimension $p^{2m} - 1 \not\equiv 0 \pmod{p}$, so its irreducible components cannot all have dimensions divisible by p . Hence the trace-zero component of $\psi^\vee \otimes \psi$ contains a subrepresentation of dimension 1; such a subrepresentation cannot be trivial, or else ψ would be forced to be reducible by Schur's lemma. Hence $\psi^\vee \otimes \psi$ contains a nontrivial subrepresentation of dimension 1, as does $\tau^\vee \otimes \tau$ since τ is completely reducible (Maschke's theorem).

□

We recall some basic theory of Tannakian categories, as introduced in [31].

Definition 2.5.2. A *Tannakian category* \mathcal{C} over K is a associative, unital, commutative K -linear tensor category with duals, admitting an exact faithful functor to the category of finite-dimensional vector spaces over some field containing K . Functors between Tannakian categories are required to respect the tensor and dual structures. Given an object $X \in \mathcal{C}$, we will write $[X]$ for the smallest subcategory of \mathcal{C} containing X and closed under formation of isomorphic objects, direct sums, tensor products, duals, and subquotients.

Let K' be a field extension of K . A *fibre functor* on \mathcal{C} with values in K' is an exact functor ω of Tannakian categories from \mathcal{C} to the category of finite-dimensional K' -vector spaces. If $K' = K$, the fibre functor is said to be *neutral*.

A \otimes -generator of \mathcal{C} is an element X such that the smallest subcategory of \mathcal{C} containing X and closed under formation of isomorphic objects, direct sums, tensor products, and subquotients (but not duals) is equal to \mathcal{C} . A Tannakian category admitting an \otimes -generator is said to be *algebraic*. For example, for any finite group G , the category of representations of G on finite-dimensional K -vector spaces is algebraic.

Lemma 2.5.3. Let \mathcal{C} be an algebraic Tannakian category over K .

- (a) There exists a fibre functor on \mathcal{C} with values in a finite extension K' of K .
- (b) The automorphism group G of any such fibre functor is finite.
- (c) In case $K' = K$, \mathcal{C} is equivalent to the category of representations of G on finite dimensional K -vector spaces.

Proof. See [31, III, Scholie 3.3.1.1] for (a), [31, III, 3.3.3(a)] for (b), and [31, III, Théorème 3.2.2] for (c). □

Lemma 2.5.4. Let $F : \mathcal{C}' \rightarrow \mathcal{C}$ be an exact functor of Tannakian categories. Let ω be a fibre functor on \mathcal{C} . Let G, G' be the automorphism groups of $\omega, \omega \circ F$, respectively.

- (a) The functor F canonically induces a map $u : G \rightarrow G'$.

- (b) Suppose that F is fully faithful, and that for every $X \in \mathcal{C}'$, every subobject of $F(X)$ is the image of a subobject of X . Then u is surjective.
- (c) Suppose that every object of \mathcal{C} is a subquotient of an object in the image of F . Then u is injective.

Proof. See [31, III, 3.3.3]. □

For the rest of the discussion of local monodromy representations, it will be convenient to set some running hypotheses.

Hypothesis 2.5.5. For the remainder of § 2.5, assume that k is algebraically closed. Let $Z \hookrightarrow X$ be a closed immersion of k -varieties, with X irreducible affine and smooth. Let P be a smooth affine formal scheme over $\text{Spf } \mathfrak{o}_K$ with $P_k \cong X$. Let v be a minimal valuation on $k(X)$ with center $z \in Z$.

Definition 2.5.6. Fix a geometric point \bar{x} of $\text{Spec } k(X)_v$. We may then identify $\pi_1(\text{Spec } k(X)_v, \bar{x})$ with a subgroup I_v of $\pi_1(X, \bar{x})$, called the *inertia subgroup* associated to v for the choice of the basepoint \bar{x} . In concrete terms, choosing \bar{x} amounts to fixing an algebraic closure $k(X)_v$, and I_v may be identified with the Galois group of that algebraic closure.

Suppose that $g : X_2 \rightarrow X$ is the normalization of the Galois closure of a separable alteration $f : X_1 \rightarrow X$ with irreducible. Then I_v surjects onto the subgroup of $\text{Aut}(X_2/X)$ fixing the center on X_2 of one of the extensions of v to $k(X_2)$. (Which center gets fixed depends on how we choose to embed $k(X_2)$ into our fixed algebraic closure of $k(X)_v$.)

Note that I_v contains a subgroup W_v (the *wild inertia subgroup*) which is a pro- p -group, such that the quotient I_v/W_v is abelian and pro-prime-to- p . See for instance [30, Chapter 6].

Definition 2.5.7. By a *localizing subspace* of P_K for v , we will mean a subspace of $]z[_P$ defined by finitely many conditions, each of the form $\epsilon \leq |g| < |h|$ for some $g, h \in \Gamma(P, \mathcal{O})$ and some $\epsilon \in [0, 1)$ such that the images $\bar{g}, \bar{h} \in \Gamma(P_k, \mathcal{O})$ of g, h satisfy $\bar{h} \neq 0$ and $\bar{g}/\bar{h} \in \mathfrak{m}_v$. Note that if we replace g, h by some other $g', h' \in \Gamma(P, \mathcal{O})$ with the same reductions, then the suprema of $|g - g'|, |h - h'|$ over P_K are strictly less than 1. For ϵ greater than or equal to these suprema, the conditions $\epsilon \leq |g| < |h|$ and $\epsilon \leq |g'| < |h'|$ are equivalent.

Remark 2.5.8. Suppose that $g, h \in \Gamma(P, \mathcal{O})$ are such that the images $\bar{g}, \bar{h} \in \Gamma(P_k, \mathcal{O})$ of g, h satisfy $\bar{h} \neq 0$ and $\bar{g}/\bar{h} \in \mathfrak{o}_v$. Then for any localizing subspace A , we can always find another localizing subspace $A' \subseteq A$ on which we have $|g| \leq |h|$ everywhere. That is because we have assumed v is minimal and k is algebraically closed, so we can find $\bar{c} \in k$ for which $\bar{g}/\bar{h} - \bar{c} \in \mathfrak{m}_v$. We can then achieve what we want by lifting \bar{c} to some $c \in \mathfrak{o}_K$ and letting A' be the subspace of A on which $|g - ch| < |h|$.

On the other hand, if one were to construct local monodromy representations without the hypotheses that v is minimal and k is algebraically closed, one would need to add conditions of the form $|g| \leq |h|$ to the definition of a localizing subspace. We will not do this here.

Note also that when constructing localizing subspaces, we will sometimes find it convenient to impose a condition of the form $\epsilon < |g| < |h|$, rather than $\epsilon \leq |g| < |h|$. This will be harmless for our purposes, because the space cut out by such a condition contains the subspace cut out by the condition $\epsilon' \leq |g| < h$ for any $\epsilon' \in (\epsilon, 1)$.

Lemma 2.5.9. *Let $f : X_1 \rightarrow X$ be a separable alteration with X_1 irreducible. Then there exist an open dense subscheme T of X and a proper morphism $\tilde{f} : P_1 \rightarrow P$ of formal schemes over $\mathrm{Spf} \mathfrak{o}_K$, such that \tilde{f} is finite étale over T and the induced map $P_{1,k} \rightarrow X$ factors through a birational morphism $P_{1,k} \rightarrow X_1$. (Beware that we do not guarantee that $P_{1,k}$ is irreducible; birationality here means that the map becomes an isomorphism over some open dense subscheme of X_1 .)*

Proof. Choose a primitive element α of $k(X_1)$ over $k(X)$ which is integral over $\Gamma(X, \mathcal{O})$. Let $\overline{Q}(t)$ be the minimal polynomial of α over $k(X)$, then lift \overline{Q} to a monic polynomial $Q(t) \in \Gamma(P, \mathcal{O})[t]$ and take $P_0 = \mathrm{Spf} \Gamma(P, \mathcal{O})[t]/(Q(t))$.

By construction, there exists a rational map $P_{0,k} \dashrightarrow X_1$. Let X'_1 be the Zariski closure of the graph of this rational map, so that X'_1 projects onto both $P_{0,k}$ and X_1 . The map $X'_1 \rightarrow P_{0,k}$ is proper birational, so it can be written as the blowup along some ideal sheaf on $P_{0,k}$ [10, Theorem 7.17]. Since $P_{0,k}$ is affine, we can lift global generators of this ideal sheaf to produce an ideal sheaf on P_0 , in which we blow up to obtain a proper morphism $P_1 \rightarrow P_0$ such that $P_{1,k}$ admits a morphism to X'_1 . This proves the claim. \square

Lemma 2.5.10. *In Lemma 2.5.9, the map $\tilde{f}_K : P_{1,K} \rightarrow P_K$ is finite étale on some strict neighborhood V of $]T[_P$. Moreover, any automorphism τ of X_1 over X of finite order lifts to an automorphism of $\tilde{f}_K^{-1}(V)$ over V , for a suitable choice of V (depending on τ).*

Proof. Set notation as in the proof of Lemma 2.5.9. The map $P_0 \rightarrow P$ is finite, so we may argue as in the proof of [35, Theorem 2.6.3(1)] to see that $P_{0,K} \rightarrow P_K$ is finite étale on some strict neighborhood V of $]T[_P$, and that a given automorphism τ of X_1 over X of finite order lifts over V . Let T_0 be the inverse image of T in $P_{0,k}$, and let V_0 be the inverse image of V in $P_{0,K}$. By Berthelot's strong fibration theorem [3, Théorème 1.3.5], the map $P_{1,K} \rightarrow P_{0,K}$ is an isomorphism over some strict neighborhood W of $]T_0[_{P_0}$. By shrinking V , we may ensure that $V_0 \subseteq W$, and then the claim follows. \square

Lemma 2.5.11. *In Lemma 2.5.9, let S be the set of centers on X_1 of valuations on $k(X_1)$ extending v . Then there exists a localizing subspace A of P_K for v such that $\tilde{f}_K^{-1}(A)$ maps into S under $P_{1,K} \xrightarrow{\mathrm{sp}} P_{1,k} \rightarrow X_1$.*

Proof. Apply Raynaud-Gruson flatification [29, première partie, §5.2] to find a blowup $X' \rightarrow X$ under which the proper transform $X'_1 \rightarrow X'$ of $X_1 \rightarrow X$ is finite flat. By blowing up P_1 as needed, we can force ourselves into the case where $P_{1,k} \rightarrow X$ factors through X'_1 .

Let z' denote the center of v on X' . Let U be an open dense affine subscheme of X' containing z' . Choose finitely many elements $g_i, h_i \in \Gamma(P, \mathcal{O})$ so that the ratios $\bar{g}_i/\bar{h}_i \in k(X)$ generate $\Gamma(U, \mathcal{O})$ as an algebra over $\Gamma(X, \mathcal{O})$. By Remark 2.5.8, we can find a localizing subspace A_1 of P_K for v on which $|g_i| \leq |h_i|$ for each i .

Next, choose finitely many elements $g_j, h_j \in \Gamma(P, \mathcal{O})$ so that the ratios $\bar{g}_j/\bar{h}_j \in k(X)$ belong to $\Gamma(U, \mathcal{O})$ and cut out a nonempty closed subscheme of U supported at z' . Let A be the subspace of A_1 on which $|g_j| < |h_j|$ for each j ; then A is a localizing subspace of P_K for v .

By construction, any point of $\tilde{f}_K^{-1}(A)$ must map to a point of $P_{1,k}$ projecting onto z' in X' . The image in X'_1 must then be the center of an extension of v , and similarly after projecting onto X_1 . This yields the claim. \square

Lemma 2.5.12. *Suppose that k is algebraically closed. Let (X, Z) be a smooth pair of k -varieties with X affine irreducible. Let P be a smooth affine formal scheme over $\mathrm{Spf} \mathfrak{o}_K$ with $P_k \cong X$, and let Q be a relative normal crossings divisor on P with $Q_k \cong Z$. Let \mathcal{E} be a convergent isocrystal on (X, Z) with nilpotent residues, realized as a log- ∇ -module on (P_K, Q_K) with nilpotent residues. Then for any closed point $x \in X$, the restriction of \mathcal{E} to $]x[_P$ is unipotent (i.e., it is a successive extension of constant ∇ -modules).*

Proof. Apply [17, Proposition 3.6.9]. \square

Remark 2.5.13. We warn the reader of a slight misuse of notation in Definition 2.5.14 below. For P a formal scheme, the notation $]S[_P$ normally indicates the tube of a subspace S of the special fibre P_k . We will also use it in settings where S is not a subspace of P_k itself, but of another scheme X to which P_k maps in a specified manner. We then mean to take the inverse image of S in P_k before forming the tube.

Definition 2.5.14. Let \mathcal{E} be an F -isocrystal on $X \setminus Z$ overconvergent along Z admitting local semistable reduction at v . Define f, X_1, Z_1, U as in Definition 2.4.1. Choose an extension v_1 of v to $k(X_1)$, let z_1 be the center of v_1 on X_1 , and let U_1 be an open dense affine subscheme of U containing z_1 . After possibly shrinking U_1 , we can construct a smooth affine formal scheme Q_1 over $\mathrm{Spf} \mathfrak{o}_K$ with $Q_{1,k} \cong U_1$, and a relative simple normal crossings divisor on Q_1 lifting $U_1 \cap Z_1$.

Let $g : X_2 \rightarrow X$ be the normalization of the Galois closure of some separable alteration factoring through f ; we may factor $g = f \circ f_1$ for $f_1 : X_2 \rightarrow X_1$ another separable alteration. Put $G = \mathrm{Aut}(X_2/X)$. Choose an extension v_2 of v_1 to $k(X_2)$, and let z_2 be the center of v_2 on X_2 ; then I_v surjects onto the stabilizer H of z_2 in G .

By Lemma 2.5.9, there exist an open dense affine subscheme T of X and a proper morphism $\tilde{h} : P_2 \rightarrow P$ of formal schemes, such that $h : P_{2,k} \rightarrow X$ factors through X_2 , and over T , \tilde{h} is finite étale and reduces to g . We may then shrink T to ensure that $T_1 = f^{-1}(T) \subseteq U_1$. Put $T_2 = f_1^{-1}(T_1), U_2 = f_1^{-1}(U_1)$.

By Lemma 2.5.10, there exists a strict neighborhood V of $]T[_P$ (see Remark 2.5.13) in P_K over which \tilde{h}_K is finite étale Galois with group G . Note that as we shrink V , $V_2 = \tilde{h}_K^{-1}(V)$ runs through a cofinal set of strict neighborhoods of $]T_2[_{P_2}$ in $P_{2,K}$. After shrinking V suitably, we may realize $\mathcal{E}|_T$ as a ∇ -module \mathcal{E}_V on V , such that $\tilde{h}_K^* \mathcal{E}_V$ realizes $(g^* \mathcal{E})|_{T_2}$ on V_2 .

On the other hand, we may realize $(f^* \mathcal{E})|_{T_1}$ as a ∇ -module \mathcal{E}_{W_1} on some strict neighborhood W_1 of $]T_1[_{Q_1}$ in $Q_{1,K}$. By [17, Theorem 6.4.1], after shrinking W_1 suitably, the ∇ -module $(f^* \mathcal{E})|_{T_1}$ extends to a log- ∇ -module \mathcal{F} over all of $Q_{1,K}$. By Lemma 2.5.12, \mathcal{F} is unipotent on $]z_1[_{Q_1}$, hence also on $W_1 \cap]z_1[_{Q_1}$.

Let P'_2 be the open formal subscheme of P_2 supported on the inverse image of U_1 under the map $P_{2,k} \rightarrow X_2 \rightarrow X_1$. Let Γ denote the graph of the morphism $P'_{2,k} \rightarrow U_1$; this coincides with the Zariski closure of T_2 in $P'_{2,k} \times U_1$ under the product of the embedding $T_2 \rightarrow P'_{2,k}$ and the composition $T_2 \rightarrow T_1 \rightarrow U_1$. In particular, it is isomorphic to $P'_{2,k}$.

We can construct two realizations of $(g^*\mathcal{E})|_{T_2}$ as ∇ -modules on a strict neighborhood W_2 of $]T_2[P'_2 \times Q_1]$ in $]\Gamma[P'_2 \times Q_1]$, one by pulling back $\tilde{h}_K^*\mathcal{E}_V$ along the first projection, the other by pulling back \mathcal{E}_{W_1} along the second projection. By the functoriality of rigid cohomology, for W_2 suitably small, there is a distinguished isomorphism between these two realizations. The second realization is unipotent on $W_2 \cap (P'_{2,K} \times]z_1[Q_1])$, hence also on $W_2 \cap]z_2[P'_2 \times Q_1]$. This implies in turn that the the first realization is unipotent on $W_2 \cap]z_2[P'_2 \times Q_1]$, so $\tilde{h}_K^*\mathcal{E}_V$ is unipotent on $V_2 \cap]z_2[P_2]$.

By applying Lemma 2.5.11, we obtain a localizing subspace A of P_K for v such that $\tilde{h}_K^{-1}(A)$ is finite étale Galois over A , and the restriction \mathcal{E}_A of \mathcal{E} to A becomes unipotent upon pullback to $B = \tilde{h}_K^{-1}(A) \cap]z_2[P_2$. The semisimplification $\mathcal{E}_A^{\text{ss}}$ then becomes constant upon pullback to B ; let $\mathcal{E}_B^{\text{ss}}$ denote this pullback.

Note that $\tilde{h}_K^{-1}(A)$ is a disjoint union of copies of B corresponding to points of X_2 in the G -orbit of z_2 , so B is finite étale Galois over A with group H . We thus obtain an action of H on $H^0(\mathcal{E}_B^{\text{ss}})$, and by composition a homomorphism $\tau : I_v \rightarrow \text{GL}(H^0(\mathcal{E}_B^{\text{ss}}))$. Note that the construction does not depend on the choice of g : if $g' : X'_2 \rightarrow X$ is the normalization of the Galois closure of another separable alteration factoring through g , the action of $\text{Aut}(X'_2/X)$ on local horizontal sections factors through G , so we end up with the same representation τ up to isomorphism. We refer to τ as the *semisimplified local monodromy representation* associated to \mathcal{E} at v .

We will need the following Tannakian interpretation of the construction of local monodromy representations.

Lemma 2.5.15. *With notation as in Definition 2.5.14, the Tannakian category $[\mathcal{E}_A^{\text{ss}}]$ of ∇ -modules on A generated by $\mathcal{E}_A^{\text{ss}}$ (not necessarily admitting Frobenius structures) is equivalent to the Tannakian category $[\tau]$ of representations of I_v on finite-dimensional K -vector spaces generated by τ . Moreover, the latter category contains all representations of the group $\tau(I_v)$ on finite-dimensional K -vector spaces, and τ is a \otimes -generator of this category; in particular, both Tannakian categories are algebraic with automorphism group $\tau(I_v)$.*

Proof. We check that τ is a \otimes -generator of $[\tau]$ by an elementary argument (it may also be deduced using Tannaka duality). Let $\chi : \tau(I_v) \rightarrow K$ be the character of τ , so that $\chi(1) = \dim(\tau)$ and $\chi(g) \neq \dim(\tau)$ for $g \in \tau(I_v) \setminus \{1\}$. Since the $\chi(g)$ are algebraic integers, we can find $P \in \mathbb{Z}[t]$ for which $P(\dim(\tau)) > 0$ but $P(\chi(g)) = 0$ for $g \in \tau(I_v) \setminus \{1\}$. Thus the virtual character $P(\chi)$ is a positive multiple of the character of the regular representation, in which each irreducible representation of $\tau(I_v)$ must appear with positive multiplicity. Write $P = Q - R$ with $Q, R \in \mathbb{Z}[t]$ having nonnegative coefficients; then $Q(\chi) + R(\chi)$ is a true character in which each irreducible representation occurs with positive multiplicity. This proves the claim.

On one hand, $[\tau]$ now equals the category of representations of $\tau(I_v)$ on finite-dimensional K -vector spaces. On the other hand, $[\tau]$ is algebraic. Let ω be the forgetful functor from $[\tau]$ to finite-dimensional K -vector spaces, and put $G = \text{Aut}(\omega)$. By Lemma 2.5.3(c), $[\tau]$ is also equivalent to the category of representations of G on finite-dimensional K -vector spaces. We obtain automorphisms of ω from elements of $\tau(I_v)$, giving an inclusion $\tau(I_v) \rightarrow$

G . This inclusion must be an isomorphism, otherwise we could construct nonisomorphic representations of G whose restrictions to $\tau(I_v)$ are isomorphic (e.g., the induction to G of the trivial representation of $\tau(I_v)$, and a direct sum of trivial representations of the same total dimension). Hence $G = \tau(I_v)$.

Let $F : [\mathcal{E}_A^{\text{ss}}] \rightarrow [\tau]$ be the faithful functor taking a ∇ -module \mathcal{F} to $H^0(\mathcal{F}_B)$, where \mathcal{F}_B denotes the pullback to B . We check that F is fully faithful. Given $\mathcal{F}_1, \mathcal{F}_2 \in [\mathcal{E}_A^{\text{ss}}]$, any I_v -equivariant morphism $H^0(\mathcal{F}_{1,B}) \rightarrow H^0(\mathcal{F}_{2,B})$ corresponds to an I_v -fixed horizontal section of $\mathcal{F}_{1,B}^\vee \otimes \mathcal{F}_{2,B}$, which then descends to a horizontal section of $\mathcal{F}_1^\vee \otimes \mathcal{F}_2$. This in turn yields a morphism $\mathcal{F}_1 \rightarrow \mathcal{F}_2$, proving the claim.

We check that F is essentially surjective. From above, every irreducible $\rho \in [\tau]$ occurs as a subobject of $\tau^{\otimes n}$ for some nonnegative integer n . Since $[\tau]$ is semisimple (Maschke's theorem), we can find an endomorphism of $\tau^{\otimes n}$ which projects onto one copy of ρ . This endomorphism corresponds to an I_v -fixed horizontal section of $(\mathcal{E}_B^{\text{ss}})^{\otimes(-n)} \otimes (\mathcal{E}_B^{\text{ss}})^{\otimes n}$, thus to a horizontal section of $(\mathcal{E}_A^{\text{ss}})^{\otimes(-n)} \otimes (\mathcal{E}_A^{\text{ss}})^{\otimes n}$ and to an endomorphism of $(\mathcal{E}_A^{\text{ss}})^{\otimes n}$. The image of this last endomorphism is an element $\mathcal{F} \in [\mathcal{E}_A^{\text{ss}}]$ for which $F(\mathcal{F}) \cong \rho$. \square

Proposition 2.5.16. *Suppose that k is algebraically closed. Let $Z \hookrightarrow X$ be a closed immersion of k -varieties, with X irreducible affine and $X \setminus Z$ smooth. Let v be a minimal valuation on $k(X)$ centered on Z . Let $f_i : X_i \rightarrow X$ be an exposing sequence for v . Let \mathcal{E} be an F -isocrystal on $X \setminus Z$ overconvergent along Z admitting local semistable reduction at v . If the semisimplified local monodromy representation associated to \mathcal{E} is trivial, then there exists an index i such that $f_i^*\mathcal{E}$ is log-extendable.*

Proof. Set notation as in Definition 2.5.14, and suppose that τ is trivial. By Lemma 2.5.15, \mathcal{E}_A is unipotent.

Recall that A was defined using finitely many conditions of the form $\epsilon \leq |g_j| < |h_j|$ with $g_j, h_j \in \Gamma(P, \mathcal{O})$, $\bar{h}_j \neq 0$, and $v(\bar{g}_j/\bar{h}_j) > 0$. In particular, we can find a blowup X' of X such that the \bar{g}_j/\bar{h}_j belong to the local ring of X' at the center z' of v . Let S be the subset of $S_{k(X)/k}$ consisting of valuations whose center on X' is equal to z' ; this is an open set for the patch topology.

We can thus choose an index i so that the divisorial valuations of $k(X_i)$ corresponding to the components of $f_i^{-1}(Z)$ all belong to S . By one direction of [17, Theorem 6.4.5], $f_i^*\mathcal{E}$ has unipotent local monodromy along each of these valuations. By the other direction of [17, Theorem 6.4.5], $f_i^*\mathcal{E}$ is log-extendable, as desired. \square

Remark 2.5.17. It would simplify some arguments in this paper to construct a local monodromy representation without semisimplification, e.g., by following the procedure in the one-dimensional case given in [13, Theorem 4.45]. Unfortunately, this is difficult in the higher-dimensional setting; the main obstruction is the lack of a good comparison between the de Rham cohomologies of the spaces A and B in Definition 2.5.14. (In the one-dimensional case, they are both open annuli.) We will not attempt to tackle this issue here.

3 Analysis of p -adic differential modules

In this section, we recall some important facts from the theory of ordinary p -adic differential equations. Our reference for these is the book [20]. We have not distinguished here between results original to [20] and results of prior origin; that is done amply in [20].

Hypothesis 3.0.1. Throughout this section, let F be a complete nonarchimedean field of characteristic 0 and residue characteristic p . (It is not necessary for F to be discretely valued.) Let I be a subinterval of $[0, +\infty)$, and let \mathcal{E} be a ∇ -module of rank $n > 0$ on the disc/annulus $A_F(I)$ with coordinate x .

Definition 3.0.2. We say \mathcal{E} is *constant* if it is spanned by horizontal sections, and *unipotent* if it is a successive extension of constant objects.

3.1 Subsidiary radii of convergence

The following definition is from [20, Definition 9.4.4].

Definition 3.1.1. For $\rho \in I$ nonzero, let F_ρ be the completion of $\Gamma(A_F(I), \mathcal{O})$ for the ρ -Gauss norm, and let \mathcal{E}_ρ be the base extension of \mathcal{E} to F_ρ . Define the *generic radius of convergence* of \mathcal{E}_ρ , denoted $R(\mathcal{E}_\rho)$, as $p^{-1/(p-1)}$ divided by the spectral norm of $\frac{d}{dx}$ on \mathcal{E}_ρ ; this quantity is always positive and never greater than ρ . Note also that

$$R(\mathcal{E}_\rho^\vee) = R(\mathcal{E}_\rho)$$

and that for \mathcal{E}' another nonzero ∇ -module on $A_F(I)$,

$$R((\mathcal{E} \otimes \mathcal{E}')_\rho) \geq \min\{R(\mathcal{E}_\rho), R(\mathcal{E}'_\rho)\}$$

with equality if the minimum is achieved only once [20, Lemma 9.4.6].

Remark 3.1.2. Suppose that $0 \in I$. If $\rho' < R(\mathcal{E}_\rho)$ for some $\rho, \rho' \in I$, then Dwork's transfer theorem [20, Theorem 9.6.1] implies that the restriction of \mathcal{E} to $A_F[0, \rho']$ is constant, so $R(\mathcal{E}_{\rho'}) = \rho'$.

The following definition is from [20, Definition 9.8.1].

Definition 3.1.3. For $\rho \in I$, let $\mathcal{E}_{\rho,1}, \dots, \mathcal{E}_{\rho,m}$ be the Jordan-Hölder constituents of \mathcal{E}_ρ . Define the *subsidiary generic radii of convergence*, or *subsidiary radii*, of \mathcal{E}_ρ as the multiset consisting of, for $i = 1, \dots, m$, the generic radius of convergence of $\mathcal{E}_{\rho,i}$ with multiplicity $\dim_{F_\rho}(\mathcal{E}_{\rho,i})$. For $s \in -\log I$, define $f_1(\mathcal{E}, s) \geq \dots \geq f_n(\mathcal{E}, s)$ to be the real numbers such that the subsidiary radii of $\mathcal{E}_{e^{-s}}$ are equal to $e^{-f_1(\mathcal{E}, s)}, \dots, e^{-f_n(\mathcal{E}, s)}$.

Theorem 3.1.4. *The functions $f_i(\mathcal{E}, s)$ for $i = 1, \dots, n$ have the following properties.*

(a) *The function $f_i(\mathcal{E}, s)$ is continuous and piecewise affine-linear, with slopes in $\frac{1}{n!}\mathbb{Z}$.*

(b) If $0 \in I$ and $f_i(\mathcal{E}, s_0) > s_0$ for some $s_0 \in -\log I$, then on each side of s_0 contained in $-\log I$, the slope of $f_1(\mathcal{E}, s) + \cdots + f_i(\mathcal{E}, s)$ in a one-sided neighborhood of s_0 is nonpositive. (By Remark 3.1.2, we have $f_i(\mathcal{E}, s) = s$ for s sufficiently large.)

(c) The function $f_1(\mathcal{E}, s) + \cdots + f_i(\mathcal{E}, s)$ is convex.

Proof. See [20, Theorem 11.3.2]. \square

Corollary 3.1.5. Suppose $I = [\alpha, \beta]$ with $\alpha < \beta$. Then there exists $\gamma \in (\alpha, \beta]$ such that for $i = 1, \dots, n$, $f_i(\mathcal{E}, s)$ is affine-linear on $[\alpha, \gamma]$.

This corollary combines naturally with the following theorem.

Theorem 3.1.6. Suppose that the interval I is open, and that the following conditions are satisfied for some $i \in \{1, \dots, n-1\}$.

(a) The function $f_1(\mathcal{E}, s) + \cdots + f_i(\mathcal{E}, s)$ is affine-linear on $-\log I$.

(b) We have $f_i(\mathcal{E}, s) > f_{i+1}(\mathcal{E}, s)$ for all $s \in -\log I$.

Then there exists a unique direct sum decomposition $\mathcal{E}_1 \oplus \mathcal{E}_2$ of ∇ -modules on $A_F(I)$ such that $\text{rank}(\mathcal{E}_1) = i$, $f_j(\mathcal{E}, s) = f_j(\mathcal{E}_1, s)$ for $j = 1, \dots, i$, and $f_j(\mathcal{E}, s) = f_{j-i}(\mathcal{E}_2, s)$ for $j = i+1, \dots, n$.

Proof. See [20, Theorem 12.4.2]. \square

The $f_i(\mathcal{E}, s)$ also satisfy the following subharmonicity property.

Theorem 3.1.7. Suppose that $I = [0, \beta]$ with $\beta > 1$. Suppose $z_1, \dots, z_m \in \mathfrak{o}_F$ have distinct images in κ_F . For $j = 1, \dots, m$, let $T_j : A_F(I) \rightarrow A_F(I)$ denote the substitution $x \mapsto x + z_j$. For $i = 1, \dots, n$, let $s'_{\infty, i}$ be the left slope of $f_1(\mathcal{E}, s) + \cdots + f_i(\mathcal{E}, s)$ at $s = 0$. For $i = 1, \dots, n$ and $j = 1, \dots, m$, let $s'_{j, i}$ be the right slope of $f_1(T_j^* \mathcal{E}, s) + \cdots + f_i(T_j^* \mathcal{E}, s)$ at $s = 0$. Then

$$s'_{\infty, i} \leq \sum_{j=1}^m s'_{j, i} \quad (i = 1, \dots, n).$$

Proof. See [20, Theorem 11.3.2(c,d)]. \square

We also use the following compatibility with tame base change.

Lemma 3.1.8. Let m be a positive integer coprime to p , and let \mathcal{F} be the pullback of \mathcal{E} along the substitution $x \mapsto x^m$. Then for all $s \in -\log I$, $f_i(\mathcal{F}, s/m) - s/m = f_i(\mathcal{E}, s) - s$. In particular, $f_i(\mathcal{F}, s/m) = s/m$ if and only if $f_i(\mathcal{E}, s) = s$.

Proof. See [20, Proposition 9.7.6]. \square

3.2 The Robba condition

Notation 3.2.1. In this subsection only, for m a positive integer, let $f_m : A_F(I^{1/m}) \rightarrow A_F(I)$ denote the F -linear map for which $f_m^*(x) = x^m$.

Definition 3.2.2. We say that \mathcal{E} satisfies the *Robba condition* if $R(\mathcal{E}_\rho) = \rho$ for all nonzero $\rho \in I$. For instance, this holds if \mathcal{E} is unipotent.

As noted in [20], the following results follow from the Christol-Mebkhout theory of p -adic exponents. However, see Remark 3.2.6 below.

Theorem 3.2.3. Suppose that I is open and that $1 \in I$. Let q be a power of p . Suppose that \mathcal{E} satisfies the Robba condition and admits an isomorphism $(\phi_K \circ f_q)^* \mathcal{E} \cong \mathcal{E}$ over $A_F(I^{1/q})$ for some isometric endomorphism $\phi_K : K \rightarrow K$. Then there exists a positive integer m coprime to p such that $f_m^* \mathcal{E}$ is unipotent.

Proof. See [20, Corollary 13.6.2]. \square

Theorem 3.2.4. Suppose that I is open, and that I' is an open subinterval of I . If \mathcal{E} satisfies the Robba condition, and the restriction of \mathcal{E} to $A_F(I')$ is constant (resp. unipotent), then so is \mathcal{E} itself.

Proof. See [20, Corollary 13.6.4]. \square

Proposition 3.2.5. Suppose that I is open and that \mathcal{E} satisfies the Robba condition. Suppose that there exist a closed subinterval J of I with nonempty interior and a finite étale cover $g : X \rightarrow A_F(J)$ such that $g^* \mathcal{E}$ is unipotent. Then there exists a positive integer m coprime to p such that $f_m^* \mathcal{E}$ is unipotent.

Proof. (Compare [20, Proposition 19.4.5], where a different proof is given.) Since unipotence can be checked after enlarging F , we may assume that F contains a p -th root of unity ζ_p . We may also assume that there exists $\rho \in |F^*|$ in the interior of J ; by rescaling, we may then reduce to the case $\rho = 1$.

Note that $\sigma = \sigma_K \circ f_p : A_F(J^{1/p}) \rightarrow A_F(J)$ extends uniquely to a map $\tilde{\sigma} : X \times_{A_F(J)} A_F(J^{1/p}) \rightarrow X$, whereas the inclusion $\iota : A_F(J^{1/p}) \rightarrow A_F(J)$ induces a second map $\tilde{\iota} : X \times_{A_F(J)} A_F(J^{1/p}) \rightarrow X$. Put $\mathcal{F} = g_* \mathcal{O}_X$. Since the differential modules $\tilde{\sigma}^* \mathcal{O}_X$ and $\tilde{\iota}^* \mathcal{O}_X$ are both trivial, they are isomorphic to each other; we thus obtain an isomorphism $F : \sigma^* \mathcal{F} \cong \iota^* \mathcal{F}$.

Since $g^* \mathcal{E}$ is unipotent, there exists a nonzero morphism $g^* \mathcal{E} \rightarrow \mathcal{O}_X$ of ∇ -modules on X . By adjunction, there also exists a nonzero morphism $\mathcal{E} \rightarrow \mathcal{F}$ over $A_F(J)$. Let \mathcal{G} be the minimal σ -stable ∇ -submodule of \mathcal{F} containing the image of this morphism; then \mathcal{G} satisfies the Robba condition. (More precisely, put $\mathcal{G}_N = \bigoplus_{i=0}^N \sigma^{i*} \mathcal{E}$ as a ∇ -module over $A_F(J^{1/p^N})$, and take $\mathcal{G} = \mathcal{G}_N$ for the minimal N such that \mathcal{G}_N and \mathcal{G}_{N+1} have the same restriction to $A_F(J^{1/p^{N+1}})$.) By Theorem 3.2.3, there exist a positive integer m coprime to p and some $\epsilon > 0$ such that the restriction of $f_m^* \mathcal{G}$ to $A_F(1 - \epsilon, 1 + \epsilon)$ is unipotent.

Consequently, the restriction of $f_m^* \mathcal{E}$ to $A_F(1 - \epsilon, 1 + \epsilon)$ is either unipotent or reducible. By induction on $\text{rank}(\mathcal{E})$, we may deduce that for some m coprime to p and some nonempty

open interval J' of I , the restriction of $f_m^*\mathcal{E}$ to $A_F(J')$ is unipotent. By Theorem 3.2.4, $f_m^*\mathcal{E}$ itself is unipotent. \square

Remark 3.2.6. The theory of indices of p -adic differential operators, developed by Christol-Mebkhout, separates naturally into two aspects: the theory of decompositions of solvable ∇ -modules, and the p -adic Fuchsian theory (i.e., the theory of p -adic exponents). Of these, the decomposition theory is the more accessible; the version of it we developed in [20] underlies most of the arguments in this paper.

By contrast, the theory of p -adic exponents seems to lie much deeper. The original development by Christol-Mebkhout is quite difficult; subsequent treatments (including one by Dwork, one that we gave in [20, Chapter 13], and one in an upcoming book of Christol) achieve some simplifications, but still rely on some delicate p -adic analytic estimates.

Consequently, it may be desirable to avoid use of p -adic exponents for purely aesthetic reasons. This can indeed be done, by virtue of the following observations.

- Theorem 3.2.3 is only used in the proof of Proposition 3.2.5.
- Theorem 3.2.4 can be obtained using a transfer theorem of Christol in place of the theory of p -adic exponents; see [20, Remark 13.7.3].
- Proposition 3.2.5 can be obtained without using the theory of p -adic exponents; see [20, Proposition 19.4.5]. That argument uses the relationship between generic radius of convergence and wild ramification of local Galois representations in positive characteristic, which can also be developed without p -adic exponents; see [13, Theorem 5.23].

3.3 Frobenius antecedents and descendants

Definition 3.3.1. Let $\phi : A_F(I) \rightarrow A_F(I^p)$ be the F -linear substitution $x \mapsto x^p$. A *Frobenius antecedent* of \mathcal{E} is a ∇ -module \mathcal{F} on $A_F(I^p)$ equipped with an isomorphism $\mathcal{E} \cong \phi^*\mathcal{F}$, such that $R(\mathcal{F}_\rho) > p^{-p/(p-1)}\rho$ for all $\rho \in I^p \setminus \{0\}$.

Theorem 3.3.2. Assume $I \neq [0, 0]$.

- (a) A Frobenius antecedent of \mathcal{E} exists if and only if $R(\mathcal{E}_\rho) > p^{-1/(p-1)}\rho$ for all $\rho \in I \setminus \{0\}$.
- (b) A Frobenius antecedent is unique if it exists.
- (c) If \mathcal{F} is the Frobenius antecedent of \mathcal{E} , then

$$f_i(\mathcal{F}, ps) = pf_i(\mathcal{E}, s) \quad (i = 1, \dots, n).$$

Proof. For the necessity of the condition in (a), see [20, Lemma 10.3.2]. For all of the other assertions, see [20, Theorems 10.4.2 and 10.4.4]. \square

There is also an “off-centered” variant.

Definition 3.3.3. Let $F\langle x \rangle$ denote the standard Tate algebra in the variable x over F , i.e., the completion of $F[x]$ for the 1-Gauss norm. For $\eta \in F$ with $|\eta| = 1$, let $\psi_\eta : F\langle x \rangle \rightarrow F\langle x \rangle$ be the substitution $x \mapsto (x + \eta)^p - \eta^p$. For $I \subseteq (p^{-1/(p-1)}, 1]$, a *Frobenius antecedent centered at η* of \mathcal{E} is a ∇ -module \mathcal{F} on $A_F(I^p)$ equipped with an isomorphism $\mathcal{E} \cong \psi_\eta^* \mathcal{F}$, such that $R(\mathcal{F}_\rho) > p^{-p/(p-1)}$ for all $\rho \in I^p$.

Theorem 3.3.4. Retain notation as in Definition 3.3.3.

- (a) A Frobenius antecedent centered at η of \mathcal{E} exists if and only if $R(\mathcal{E}_\rho) > p^{-1/(p-1)}$ for all $\rho \in I$.
- (b) A Frobenius antecedent centered at η is unique if it exists.
- (c) If \mathcal{F} is the Frobenius antecedent centered at η of \mathcal{E} , then

$$f_i(\mathcal{F}, ps) = pf_i(\mathcal{E}, s) \quad (i = 1, \dots, n).$$

Proof. See [20, Theorem 10.8.2]. □

Note that Frobenius descendants also come in normal and off-centered variants; we only use the off-centered version here.

Definition 3.3.5. For $I = [0, 1)$, the *Frobenius descendant centered at η* of \mathcal{E} is the pushforward $(\psi_\eta)_* \mathcal{E}$; it again may be naturally viewed as a ∇ -module on $A_F[0, 1)$, of rank $p \operatorname{rank}(\mathcal{E})$.

Theorem 3.3.6. Pick $\rho \in (p^{-1/(p-1)}, 1)$, and let s_1, \dots, s_n be the subsidiary radii of \mathcal{E}_ρ . Then the subsidiary radii of $((\psi_\eta)_* \mathcal{E})_{\rho^p}$ consist of

$$\bigcup_{i=1}^n \begin{cases} \{s_i^p, p^{-p/(p-1)} \text{ (}p-1\text{ times)}\} & s_i > p^{-1/(p-1)} \\ \{p^{-1}s_i \text{ (}p\text{ times)}\} & s_i \leq p^{-1/(p-1)}. \end{cases}$$

Proof. See [20, Theorem 10.8.3]. □

Corollary 3.3.7. For $r \in (0, \frac{1}{p-1} \log p)$, if $f_i(\mathcal{E}, r) < \frac{1}{p-1} \log p$, then $f_{i+(p-1)n}((\psi_\eta)_* \mathcal{E}, pr) = pf_i(\mathcal{E}, r)$ and

$$\sum_{j=1}^{i+(p-1)n} f_i((\psi_\eta)_* \mathcal{E}, pr) = pn \log p + p \sum_{j=1}^i f_i(\mathcal{E}, r).$$

3.4 Cyclic vectors

Definition 3.4.1. A *cyclic vector* of \mathcal{E} is an element $\mathbf{v} \in \Gamma(A_F(I), \mathcal{E})$ such that $\mathbf{v}, \frac{d}{dx}\mathbf{v}, \dots, \frac{d^{n-1}}{dx^{n-1}}\mathbf{v}$ are linearly independent (but need not form a basis). Such an element always exists; see [20, Theorem 5.7.3].

Theorem 3.4.2. *Let \mathbf{v} be a cyclic vector of \mathcal{E} , and choose $P_n, \dots, P_0 \in \Gamma(A_F(I), \mathcal{O})$ not all zero such that $P_n \frac{d^n}{dx^n} \mathbf{v} + \dots + P_0 \mathbf{v} = 0$. Pick $\rho \in I$, form the Newton polygon of the polynomial $P = P_n T^n + \dots + P_0$ measured using $|\cdot|_\rho$, and let $r_1 \leq \dots \leq r_m$ be the slopes of the polygon which are less than $\log \rho$. Then*

$$f_i(\mathcal{E}, -\log \rho) = \frac{1}{p-1} \log p - r_i \quad (i = 1, \dots, m);$$

moreover, if $m < n$, then $f_{m+1}(\mathcal{E}, -\log \rho) \leq \frac{1}{p-1} \log p - \log \rho$.

Proof. See [20, Corollary 6.5.4]. \square

3.5 Dwork modules

We introduce an example of Dwork which we will use crucially.

Lemma 3.5.1. *Assume that F contains an element π with $\pi^{p-1} = -p$ (in other words, F contains a primitive p -th root of unity). Let \mathcal{E} be the ∇ -module on $A_F(I)$ which is free on one generator \mathbf{v} satisfying*

$$\nabla(\mathbf{v}) = \pi r \mathbf{v} \otimes dx$$

for $r = \sum_{j \in \mathbb{Z}} r_j x^j \in \Gamma(A_F(I), \mathcal{O})$. Suppose that $r_j = 0$ whenever $j+1$ is divisible by p . Then

$$R(\mathcal{E}_\rho) = \min\{\rho, \min_j\{|r_j|^{-1} \rho^{-j}\}\} \quad (\rho \in I \setminus \{0\}). \quad (3.5.1.1)$$

Moreover, if $\rho < |r_j|^{-1} \rho^{-j}$ for all $\rho \in I \setminus \{0\}$ and all $j \in \mathbb{Z}$, then \mathcal{E} is constant.

Proof. If $r = r_0$, it is straightforward to check that $R(\mathcal{E}_\rho) = \min\{\rho, |r_0|^{-1}\}$ from the convergence behavior of the exponential series [20, Example 9.3.5]. Similarly, if $r = \sum_j r_j x^j$ with $\rho < |r_j|^{-1} \rho^{-j}$ for all j , then \mathcal{E}_ρ is constant because the relevant exponential converges, so $R(\mathcal{E}_\rho) = \rho$.

If $r = r_j x^j$ with $j+1$ not divisible by p , we may infer $R(\mathcal{E}_\rho) = \min\{\rho, |r_j|^{-1} \rho^{-j}\}$ by writing \mathcal{E} as the pullback along the map $x \mapsto x^{j+1}$ of the ∇ -module on $A_F(I^{j+1})$ free on one generator \mathbf{v} satisfying $\nabla(\mathbf{v}) = (j+1)^{-1} \pi r_j \mathbf{v} \otimes dx$, and using a tame base change argument (Lemma 3.1.8).

In the general case, we may assume $I = [\rho, \rho]$. We may write \mathcal{E} as a tensor product of modules, one corresponding to each term $r_j x^j$ of r for which $\rho \geq |r_j|^{-1} \rho^{-j}$, and one corresponding to the remaining terms. By the last comment in Definition 3.1.1, we obtain (3.5.1.1) for all ρ for which no two of the terms $|r_j|^{-1} \rho^{-j}$ coincide and are less than ρ . This excludes only a discrete set of values of ρ , to which we may extend (3.5.1.1) by the continuity aspect of Theorem 3.1.4. \square

4 Relative analysis

In this section, we analyze differential modules on discs defined over rings which themselves carry multiple norms. A typical such ring would be the ring of functions on an annulus; however, for the basic analysis, it will be more convenient to use a slightly more abstract class of rings. The relevance of this analysis will become more clear when we specialize to a more overtly geometric setting in the next section.

4.1 Analytic rings

We recall the setup of [14, §2], although with a different purpose in mind.

Definition 4.1.1. Let L be a complete extension of K with the same value group as K and having perfect residue field ℓ . Then for any uniformizer π of K , each $a \in L$ has a unique representation as a sum $\sum_{i=m}^{\infty} \pi^i [\overline{a_i}]$ with $\overline{a_i} \in \ell$, where $[\cdot]$ denotes the Teichmüller lift. For v a real valuation on ℓ and $\rho \in (0, 1)$, define

$$v_{n,\rho}(a) = \sup_{i \leq n} \{|\pi|^i \rho^{v(\overline{a_i})}\};$$

this function does not depend on the choice of π . For $\epsilon \in (0, 1)$, let $L_{v,\epsilon}$ be the set of $a = \sum_{i=m}^{\infty} \pi^i [\overline{a_i}] \in L$ such that $|\pi|^i \epsilon^{v(\overline{a_i})} \rightarrow 0$ as $i \rightarrow \infty$. For $a \in L_{v,\epsilon}$ and $\rho \in [\epsilon, 1)$, define

$$|a|_{\rho^v} = \lim_{n \rightarrow \infty} v_{n,\rho}(a) = \sup_{i: \overline{a_i} \neq 0} \{|\pi|^i \rho^{v(\overline{a_i})}\};$$

this is a norm on $L_{v,\epsilon}$. For $I \subseteq (0, 1)$ an interval, pick any $\epsilon \in (0, 1)$ such that $I \subseteq [\epsilon, 1)$, and let $L_{v,I}$ be the Fréchet completion of $L_{v,\epsilon}$ for the norms $|\cdot|_{\rho^v}$ for $\rho \in I$; the result is independent of ϵ . Note that for $\rho \in I$ and $n \in \mathbb{Z}$, the functions $v_{n,\rho}$ also extend continuously to $L_{v,I}$.

Remark 4.1.2. For $\epsilon \in (0, 1)$, the ring $L_{v,\epsilon}$ is a principal ideal domain [14, Proposition 2.6.5]. For $I \subseteq (0, 1)$ a closed interval, the ring $L_{v,I}$ is also a principal ideal domain [14, Proposition 2.6.8]. Both of these rely crucially on the fact that K is discretely valued. It is shown in [14, Theorem 2.9.6] for $I = [\epsilon, 1)$, and likely true for arbitrary I , that $L_{v,I}$ is a Bézout domain (a domain in which every finitely generated ideal is principal); we will not need this.

Remark 4.1.3. Beware that, as per the errata to [14] (refuting a claim in [14, §2.5]), elements of $L_{v,I}$ cannot necessarily be written as doubly infinite sums $\sum_{i \in \mathbb{Z}} \pi^i [\overline{a_i}]$.

4.2 Adding a second direction

We now add a second direction to the rings of the previous subsection.

Definition 4.2.1. For $I \subseteq (0, 1)$ an interval, let $R_{v,I}$ be the ring of series $b = \sum_{i=0}^{\infty} b_i x^i \in L_{v,I}[[x]]$ such that for each $\rho \in I$, $|b_i|_{\rho^v} \rightarrow 0$ as $i \rightarrow \infty$. For $\rho \in I$ and $s \geq 0$, define the norm $|\cdot|_{\rho^v,s}$ on $R_{v,I}$ by the formula

$$|b|_{\rho^v,s} = \sup_i \{|b_i|_{\rho^v} \rho^{is}\};$$

note that for fixed ρ , $|b|_{\rho^v,s}$ is a nonincreasing function of s .

By computing termwise, we deduce the following lemma [17, Lemma 3.1.6].

Lemma 4.2.2. For $\rho_1, \rho_2 \in I$, $s_1, s_2 \geq 0$, and $c \in [0, 1]$, put $\rho = \rho_1^c \rho_2^{1-c}$ and $s = cs_1 + (1 - c)s_2$. Then for any $b \in R_{v,I}$,

$$|b|_{\rho^v,s} \leq |b|_{\rho_1^v,s_1}^c |b|_{\rho_2^v,s_2}^{1-c}.$$

Definition 4.2.3. For $I \subseteq (0, 1)$ an interval, let $I_1 \subseteq I_2 \subseteq \dots$ be an increasing sequence of closed intervals with union I . For this definition only, put $R_i = R_{v,I_i}$. Let $C_{v,I}$ be the category whose elements are sequences M_1, M_2, \dots with M_i a finite locally free ∇ -module over R_i , equipped with isomorphisms $M_{i+1} \otimes_{R_{i+1}} R_i \cong M_i$. Note that this category is canonically independent of the choice of the I_i . Let $C_{v,*}$ denote the direct limit of the categories $C_{[\epsilon, 1]}$ as $\epsilon \rightarrow 1^-$.

Definition 4.2.4. For $M \in C_{v,I}$ and $\rho \in I$, let F_ρ be the completion of $\text{Frac } L_{v,[\rho,\rho]}$ under $|\cdot|_{\rho^v}$. We may then base extend M to obtain a ∇ -module M_ρ over the Tate algebra $F_\rho\langle x \rangle$, or in other words, a ∇ -module on the disc $A_{F_\rho}[0, 1]$. We call M_ρ the *cross section* of M at ρ . For $s \geq 0$, put $R(M, \rho, s) = R((M_\rho)_{\rho^s})$ and $f_i(M, r, s) = f_i(M_{e^{-r}}, rs)$ according to Definition 3.1.3.

4.3 Passage to the generic fibre

Definition 4.3.1. Let v' be the 0-Gauss valuation on $\ell(x)$ with respect to the valuation v on ℓ . Let ℓ' be the completion of $\ell(x)$ under v' . Put

$$\tilde{L}' = W((\ell')^\text{perf}) \otimes_{W(k)} K.$$

Let L' be the closure of $L(x)$ inside \tilde{L}' for the p -adic topology; it is a complete discretely valued field with residue field $\ell(x)$.

Remark 4.3.2. Although L' does not have perfect residue field, it fits into the formalism of [14, §2.2] using the Frobenius lift $x \mapsto x^q$. We may thus define $L'_{v',I}$ as in Definition 4.1.1, and for I closed, the result is again a principal ideal domain by [14, Proposition 2.6.8]. Moreover, given $M \in C_{v,I}$ for I closed, we can form a base extension M' to $L'_{v',I}$, which will then be a finite free $L'_{v',I}$ -module.

We will limit our use of this construction to the following key results, which are similar (and similarly proved) to Theorem 3.1.4 and Theorem 3.1.6, respectively.

Lemma 4.3.3. *Take $M \in C_{v,I}$ of rank n . For $i = 1, \dots, n$, for $s \geq 0$ fixed, the function*

$$r \mapsto f_1(M, r, s) + \dots + f_i(M, r, s)$$

is convex.

Proof. There is no harm in assuming I is closed. Also, we may enlarge ℓ to contain an element \bar{a} with $v(\bar{a}) = s$, and then pull back along the map $x \mapsto x[\bar{a}]$ to reduce to the case $s = 0$. We may now replace M by its base extension M' to $L'_{v',I}$; since this module is free, we may imitate the proof of [20, Theorem 11.3.2] to obtain the desired result.

Let us explain a bit more explicitly how this works. (For $i = 1$, one can instead proceed as in [19, Proposition 4.2.6].) It suffices to prove that each $c > 0$, the function $r \mapsto \sum_{j=1}^i \max\{f_j(M_{e^{-r}}, 0), c\}$ is convex. To check this, it suffices to do so for r in a neighborhood of any given r_0 in the interior of $-\log I$. By rescaling, we may assume $r_0 \in (0, \frac{1}{p-1} \log p)$; then by replacing M by a Frobenius descendant and using Theorem 3.3.6, we may reduce the case for a given c to the case for pc . Repeating this construction finitely many times, we may force $c > \frac{1}{p-1} \log p$.

We may treat this case by following [20, Lemma 11.5.1]. (For an even more similar argument, see [23, Theorem 2.2.6(d)].) Namely, the argument in the proof of [20, Lemma 11.5.1] shows that there is a basis of M' on which $\frac{d}{dx}$ acts via a matrix N such that, for r in a neighborhood of r_0 , the multiset of norms under $|\cdot|_{e^{-r}}$ of the eigenvalues of N coincides with the multiset of the $p^{-1/(p-1)} e^{f_j(M_{e^{-r}}, 0)}$ in their values greater than e^c . We can thus read off the desired convexity from the Newton polygon of the characteristic polynomial of N , i.e., by following [20, Theorem 11.2.1] (and [20, Remark 11.2.4]) but using Lemma 4.2.2 in place of [20, Proposition 8.2.3(b)]. \square

Lemma 4.3.4. *Take $M \in C_{v,I}$ of rank n . Suppose that for $i = 1, \dots, n$, the function $f_i(M, r, 0)$ is affine. Then for any closed subinterval I' of the interior of I , there exists a direct sum decomposition $M' \otimes_{L'_{v,I}} L'_{v,I'} \cong \bigoplus_j M'_j$ such that for each j , the functions $f_i(M'_j, r, 0)$ are the same for all i . (This decomposition becomes unique if we insist that the functions $f_i(M'_j, r, 0)$ be distinct for distinct j .)*

Proof. In the notation of the proof of Lemma 4.3.3, this follows as in the proof of [20, Theorem 12.2.2] (or more exactly [23, Theorem 2.3.5]). \square

4.4 Frobenius structures on ∇ -modules

Definition 4.4.1. Let σ_L be the unique q -power Frobenius lift on L extending σ_K ; it acts by

$$\sigma_L(a) = \sum_{i=m}^{\infty} \sigma_K(\pi)^i [\bar{a}_i^q]$$

and satisfies

$$v_{n,\rho}(a) = v_{n,\rho^{1/q}}(\sigma_L(a)).$$

One consequence is that σ_L carries $L_{v,\epsilon}$ to $L_{v,\epsilon^{1/p}}$; another is that for $a \in L_{v,\epsilon}$ and $\rho \in [\epsilon, 1)$, $|a|_{\rho^v} = |\sigma_L(a)|_{\rho^{v/q}}$. (Here we write $\rho^{v/q}$ as shorthand for $(\rho^{1/q})^v$.) As a result, σ_L extends continuously to $L_{v,I}$ and carries that ring into $L_{v,I^{1/q}}$.

Definition 4.4.2. By a *Frobenius lift* on $R_{v,[\epsilon,1)}$ relative to σ_L , we will mean a map $\sigma : R_{v,[\epsilon,1)} \rightarrow R_{v,[\epsilon^{1/q},1)}$ of the form

$$\sigma \left(\sum_i b_i x^i \right) = \sum_i \sigma_L(b_i) \sigma(x)^i$$

for which there exists $\lambda \in (0, 1)$ such that $|\sigma(x) - x^q|_{\rho^{v/q},0} \leq \lambda$ for $\rho \in [\epsilon, 1)$. For $b \in R_{v,[\epsilon,1)}$ and $\rho \in [\epsilon, 1)$, we have $|b|_{\rho^v,0} = |\sigma(b)|_{\rho^{v/q},0}$. (If we also have $\sigma(x) \equiv 0 \pmod{x^q}$, then we also have $|\sigma(x) - x^q|_{\rho^{v/q},s} \leq \lambda |x^q|_{\rho^{v/q},s}$ for all $s \geq 0$.) We call σ the *standard Frobenius lift* if $\sigma(x) = x^q$.

Definition 4.4.3. Let $\sigma : R_{v,[\epsilon,1)} \rightarrow R_{v,[\epsilon^{1/q},1)}$ be a Frobenius lift. For $M \in C_{v,[\epsilon,1)}$, a *Frobenius structure* on M with respect to σ is an isomorphism $F : \sigma^* M \cong M$ of objects in $C_{v,[\epsilon^{1/q},1)}$.

Lemma 4.4.4. For $M \in C_{v,[\epsilon,1)}$ admitting a Frobenius structure, $R(M, \rho, 0) \rightarrow 1$ as $\rho \rightarrow 1^-$.

Proof. Put $I_m = [\epsilon^{1/q^m}, \epsilon^{1/q^{m+1}}]$ for $m = 0, 1, \dots$. Choose a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of $M \otimes R_{v,I_0}$, on which $\frac{d}{dx}$ acts via the matrix N . Then the action on $F(\mathbf{e}_1), \dots, F(\mathbf{e}_n)$ is via the matrix $(d\sigma(x)/dx)\sigma(N)$. Hence for any $\delta > 0$, we can find m_0 such that for $m \geq m_0$, the matrix of action N_m of $\frac{d}{dx}$ on the basis $F^m(\mathbf{e}_1), \dots, F^m(\mathbf{e}_n)$ of $M \otimes R_{v,I_m}$ satisfies $|N_m|_{\rho^v,0} < \delta$ for $\rho \in I_m$. As in the proof of [13, Lemma 6.4], this implies the claim. \square

Lemma 4.4.5. Let σ_1 and σ_2 be two Frobenius lifts on $R_{v,[\epsilon,1)}$, let $M \in C_{v,[\epsilon,1)}$, and let $F_1 : \sigma_1^* M \cong M$ be a Frobenius structure with respect to σ_1 . Then there exists $\eta \in [\epsilon, 1)$ such that for $I \subseteq [\eta, 1)$ closed and $\mathbf{v} \in M \otimes R_{v,I}$, the series

$$F_2(\mathbf{v}) = \sum_{i=0}^{\infty} \frac{(\sigma_2(x) - \sigma_1(x))^i}{i!} F_1 \left(\frac{d^i}{dx^i}(\mathbf{v}) \right)$$

converges in $M \otimes R_{v,I^{1/q}}$. For such η , F_2 is a Frobenius structure with respect to σ_2 on the restriction of M to $C_{v,[\eta,1)}$.

Proof. By Lemma 4.4.4, there exists $\eta \in (0, 1)$ such that $R(M, \rho, 0) > \lambda \geq |\sigma_2(x) - \sigma_1(x)|_{\rho^v,0}$ for $\rho \in [\eta, 1)$. For such η , we obtain the desired convergence. \square

4.5 Behavior at the boundary

Lemma 4.5.1. Suppose $M \in C_{v,[\epsilon,1)}$ is of rank n and admits a Frobenius structure for the standard Frobenius lift σ . If $r_0 \in (0, -\log \epsilon]$ and $s \geq 0$ satisfy $f_1(M, r_0, s) < \frac{p}{p-1} \log p + r_0 s$, then there exist constants $b_i(M, s)$ for $i = 1, \dots, n$ such that for $r \in (0, r_0]$, $f_i(M, r, s) = b_i(M, s)r$.

Proof. Define $b_i(M, s) = f_i(M, r_0, s)/r_0$. By Theorem 3.3.2, for each nonnegative integer m , $f_i(M, r_0/q^{m+1}, s) = f_i(M, r_0/q^m, s)/q$ for $i = 1, \dots, n$. Consequently, the function $f_1(M, r, s) + \dots + f_i(M, r, s)$, which is convex by Lemma 4.3.3, agrees with the linear function $(b_1(M, s) + \dots + b_i(M, s))r$ for $r = r_0/q^m$ for each nonnegative integer m . On each interval $[r_0/q^{m+2}, r_0/q^m]$, we now have a convex function which agrees with an affine function at both endpoints plus one interior point, so the two functions must agree identically. That is, $f_1(M, r, s) + \dots + f_i(M, r, s) = (b_1(M, s) + \dots + b_i(M, s))r$ for $r_0/q^{m+2} \leq r \leq r_0/q^m$ for each nonnegative integer m . This yields the desired result. \square

Corollary 4.5.2. *Suppose $M \in C_{v, [\epsilon, 1]}$ is of rank n and admits a Frobenius structure. Then there exist $r_0 \in (0, -\log \epsilon]$ and constants $b_i(M, s)$ for $i = 1, \dots, n$ such that for $r \in (0, r_0]$ and $s \geq 0$, $f_i(M, r, s) = b_i(M, s)r$.*

Proof. By Lemma 4.4.5, we may change to the standard Frobenius lift σ , at the expense of possibly changing ϵ . By Lemma 4.4.4, we have $f_1(M, r, 0) \rightarrow 0$ as $r \rightarrow 0^+$; thus for r sufficiently small, we have

$$\frac{p}{p-1} \log p + rs > f_1(M, r, 0) + rs \geq f_1(M, r, s).$$

(The second inequality above follows from Theorem 3.1.4, which implies that $f_1(M, r, s)$ is convex and identically equal to rs for s large; hence the slope of any secant line of the graph of $f_1(M, r, s)$ is at most rs .) We may then apply Lemma 4.5.1. \square

Definition 4.5.3. For $I \subseteq [\epsilon, 1)$ and $h \in L_{v, \epsilon}$ with $|h|_{\rho^v} \leq 1$ for $\rho \in I$, let $T_h : R_{v, I} \rightarrow R_{v, I}$ be the translation $\sum_{i=0}^{\infty} b_i x^i \mapsto \sum_{i=0}^{\infty} b_i (x + h)^i$. For any $b \in R_{v, I}$, $\rho \in I$, $s \geq 0$ such that $\rho^s \geq |h|_{\rho^v}$, we have $|b|_{\rho^v, s} = |T_h(b)|_{\rho^v, s}$; in particular, this always holds for $s = 0$.

Lemma 4.5.4. *For $M \in C_{v, [\epsilon, 1)}$ of rank n , for $h \in L_{v, \epsilon}$ with $|h|_L \leq 1$ and $v(\bar{h}) \geq 0$, and $s \in [0, v(\bar{h})]$, we have $b_i(M, s) = b_i(T_h^* M, s)$ for $i = 1, \dots, n$.*

Proof. Suppose first that $\bar{h} \neq 0$, or equivalently $|h|_L = 1$. We then have $|h|_{\rho^v} = \rho^{v(\bar{h})}$ for ρ sufficiently close to 1. For such ρ , if $s \in [0, v(\bar{h})]$, then $|b|_{\rho^v, s} = |T_h(b)|_{\rho^v, s}$ for all $b \in R_{v, [\rho, \rho]}$. The claim follows at once.

Suppose next that $\bar{h} = 0$, or equivalently $|h|_L < 1$. In this case, $|h|_{\rho^v}$ tends to $|h|_L$ as ρ tends to 1. Consequently, for each $s \geq 0$, for ρ sufficiently close to 1, we have $\rho^s \geq |h|_{\rho^v}$; for such ρ , again $|b|_{\rho^v, s} = |T_h(b)|_{\rho^v, s}$ for all $b \in R_{v, [\rho, \rho]}$, and the claim follows. \square

Remark 4.5.5. The value of $b_i(M, s)$ is impervious to enlargement of ℓ .

4.6 Variation along a path

Hypothesis 4.6.1. Through the end of Subsection 4.8, suppose that ℓ is complete under v and algebraically closed. Let $\ell\langle x \rangle$ denote the standard Tate algebra in x over ℓ . Let $M \in C_{v, [\epsilon, 1)}$ be an object of rank n admitting a Frobenius structure.

Definition 4.6.2. Let $\alpha \in \mathbb{D}_\ell$ be a point of type (ii) or (iii). By Proposition 2.2.7, we may represent $\alpha = \alpha_{\bar{h}, r}$ for some $\bar{h} \in \mathfrak{o}_\ell$ and $r = r(\alpha) \in (0, 1]$. Choose $h \in L_{v,\epsilon}$ with $|h|_L \leq 1$ lifting \bar{h} , and define $b_i(M, \alpha) = b_i(T_h^*M, -\log r(\alpha))$ for $i = 1, \dots, n$. By Lemma 4.5.4, this definition does not depend on the choice either of h or of \bar{h} . (In particular, one may choose $h = [\bar{h}]$.) Moreover, as in Remark 4.5.5, we may replace ℓ by any complete extension without changing the $b_i(M, \alpha)$. In particular, by enlarging ℓ , we may convert points of type (iv) into points of type (ii) or (iii), and so we may define $b_i(M, \alpha)$ for such α also.

Definition 4.6.3. For $\alpha \in \mathbb{D}_\ell$, for $s \in [0, -\log r(\alpha)]$ (or $s \in [0, +\infty)$ if $r(\alpha) = 0$) and $i = 1, \dots, n$, define $b_i(M, \alpha, s)$ to be the value $b_i(M, \beta)$, for β the point of radius e^{-s} on the generic path to α .

Proposition 4.6.4. For fixed M, α , the functions $b_i(M, \alpha, s)$ for $i = 1, \dots, n$ have the following properties.

- (a) Each $b_i(M, \alpha, s)$ is continuous and piecewise affine-linear on $[0, -\log r(\alpha)]$ (or on $[0, +\infty)$ if $r(\alpha) = 0$) with slopes in $\frac{1}{n!}\mathbb{Z}$.
- (b) The function $b_1(M, \alpha, s) + \dots + b_n(M, \alpha, s)$ is convex, and is nonincreasing in a neighborhood of any point where $b_i(M, \alpha, s) > s$. (If $b_1(M, \alpha, s_0) = s_0$, then $b_1(M, \alpha, s) = s$ for $s \geq s_0$ by Remark 3.1.2.)

Proof. Again, we may assume that α is of type (ii) or (iii). By Lemma 4.4.5, M admits a Frobenius structure for the standard Frobenius lift; this induces a Frobenius structure on T_h^*M for the Frobenius lift

$$x \mapsto (x - h)^q + h^q.$$

Thus by applying a suitable T_h^* , we may reduce to the case where $\alpha \geq \alpha_{0,0}$. By Corollary 4.5.2, we can choose $r_0 > 0$ such that $f_i(M, r, s) = b_i(M, s)r$ for $r \in (0, r_0]$ and $s \geq 0$. We thus deduce the claims by applying Theorem 3.1.4 to the cross section $M_{e^{-r_0}}$. \square

Definition 4.6.5. If $r(\alpha) < 1$, then Proposition 4.6.4 implies that for some $s_0 \in (0, -\log r(\alpha))$, each function $b_i(M, \alpha, s)$ is affine-linear on $[s_0, -\log r(\alpha)]$. We call the slope of $b_i(M, \alpha, s)$ in this range the *terminal slope* of $b_i(M, \alpha, s)$.

Remark 4.6.6. For α of type (i) (i.e., $r(\alpha) = 0$), the function $b_1(M, \alpha, s) - s$ has slope at most -1 as long as its value is positive, so it must at some point s_0 achieve the value 0 . By Remark 3.1.2, we have $b_1(M, \alpha, s) = s$ for any $s > s_0$. Since $b_1(M, \alpha, s) \geq b_i(M, \alpha, s) \geq s$ for all s , we also have $b_i(M, \alpha, s) = s$ for $i = 1, \dots, n$ and $s \geq s_0$. That is, all of the terminal slopes are equal to 1 .

4.7 More cross-sectional analysis

Throughout this subsection, retain Hypothesis 4.6.1. We wish to use the cross section M_ρ to make a closer analysis of the $b_i(M, \alpha, s)$ that is peculiar to the case where α is of type (iv). To do so, we must verify that under certain explicit conditions, we can make a uniform

choice of ρ such that M_ρ can be used to recover the $b_i(M, \alpha, s)$ for all s . Lemma 4.5.1 does this for $\alpha = \alpha_{0,0}$, but we cannot uniformly translate into this case without converting α to a type other than (iv).

We first show that in certain ranges, the norm $|T_{[\bar{h}]}(\cdot)|_{\rho^v, s}$ can be written in terms of Teichmüller lifts.

Lemma 4.7.1. *Let σ be the standard Frobenius lift. Suppose $\bar{h} \in \mathfrak{o}_\ell^*$, $\rho \in [\epsilon, 1)$, and*

$$0 \leq s \leq \frac{\log p}{(-\log \rho)}.$$

For $b = \sum_{i=0}^{\infty} b_i x^i \in (\mathfrak{o}_L[[x]] \otimes_{\mathfrak{o}_L} L) \cap F_\rho \langle x \rangle$, write $b = \sum_{j=m}^{\infty} \pi^j [\bar{b}_j]$. Then

$$|T_{[\bar{h}]}(b)|_{\rho^v, s} = \sup_j \{|\pi|^j \rho^{v_s(\bar{b}_j)}\}, \quad (4.7.1.1)$$

where v_s is the s -Gauss valuation on $\ell \langle \bar{x} - \bar{h} \rangle$ (i.e., in the variable $\bar{x} - \bar{h}$).

Note that while σ does not appear in the following proof, the choice of σ affects the expression of b as a sum of Teichmüller elements. In particular, making σ standard forces $x = [\bar{x}]$.

Proof. Let $|\cdot|_1$ and $|\cdot|_2$ be the norms defined by the left and right sides of (4.7.1.1), respectively. For any fixed b , both $|b|_1$ and $|b|_2$ vary continuously in s . Consequently, we may assume $s < (\log p)/(-\log \rho)$ hereafter.

We first note that

$$x - [\bar{h}] = [\bar{x} - \bar{h}] + \sum_{i=1}^{\infty} p^i P_i(x^{1/p^i}, [\bar{h}^{1/p^i}]),$$

where P_i is homogeneous of degree p^i with coefficients in \mathbb{Z} . Since $|x|_2, |[\bar{h}]|_2 \leq 1$ and $s < (\log p)/(-\log \rho)$, this implies

$$|x - [\bar{h}] - [\bar{x} - \bar{h}]|_2 \leq p^{-1} < \rho^s = |[\bar{x} - \bar{h}]|_2 = |x - [\bar{h}]|_1.$$

In particular, $|x - [\bar{h}]|_2 = |x - [\bar{h}]|_1 = \rho^s$. For $i \geq 0$, we have

$$\begin{aligned} |(x - [\bar{h}])^i - [\bar{x} - \bar{h}]^i|_2 &= |x - [\bar{h}] - [\bar{x} - \bar{h}]|_2 \left| \frac{(x - [\bar{h}])^i - [\bar{x} - \bar{h}]^i}{x - [\bar{h}] - [\bar{x} - \bar{h}]} \right|_2 \\ &\leq p^{-1} \rho^{s(i-1)} < |(x - [\bar{h}])^i|_1. \end{aligned}$$

To deduce (4.7.1.1) for a given b , write $b = \sum_{i=0}^{\infty} b_i (x - [\bar{h}])^i$ with $b_i \in L$. For each $s \in [0, (\log p)/(-\log \rho))$ for which there is a unique value i maximizing $|b_i (x - [\bar{h}])^i|_1$, we have

$$\begin{aligned} |b|_1 &= |b_i (x - [\bar{h}])^i|_1 \\ &= |b_i [\bar{x} - \bar{h}]^i|_2 \\ |b - b_i [\bar{x} - \bar{h}]^i|_2 &< |b_i (x - [\bar{h}])^i|_1. \end{aligned}$$

Hence $|b|_1 = |b|_2$ for such s . Since this excludes only countably many s , we may deduce the claim for all s by continuity. \square

Corollary 4.7.2. *For σ, \bar{h}, ρ, s as in Lemma 4.7.1, for any $b \in F_\rho \langle x \rangle$,*

$$|T_{[\bar{h}]}(b)|_{\rho^v, s} = |T_{[\bar{h}]}(\sigma(b))|_{\rho^{v/q}, s}.$$

Proof. For $b \in L_{v, [\rho, \rho]} \langle x \rangle$, this follows from Lemma 4.7.1 and the fact that $(\mathfrak{o}_L[[x]] \otimes_{\mathfrak{o}_L} L) \cap F_\rho \langle x \rangle$ is dense in $L_{v, [\rho, \rho]} \langle x \rangle$. The claim for $b \in F_\rho \langle x \rangle$ follows immediately thereafter. \square

Corollary 4.7.3. *For σ, \bar{h}, ρ, s as in Lemma 4.7.1, put $r = -\log \rho$. If $f_1(M, r, 0) < \frac{p}{p-1} \log p$, then*

$$f_i(T_{[\bar{h}]}^* M, r/q, s) = f_i(T_{[\bar{h}]}^* M, r, s)/q \quad (i = 1, \dots, n).$$

Hence if $\alpha_{\bar{h}, e^{-s}} \geq \alpha$, then

$$f_i(T_{[\bar{h}]}^* M, r, s) = b_i(M, \alpha, s)r \quad (i = 1, \dots, n).$$

Proof. Before proceeding, we observe (as in Proposition 4.6.4) that the Frobenius structure on M for the standard Frobenius lift induces a Frobenius structure on $T_{[\bar{h}]}^* M$ for the Frobenius lift

$$x \mapsto (x - [\bar{h}])^q + [\bar{h}]^q.$$

Consequently, each Jordan-Hölder factor of $T_{[\bar{h}]}^* M$ admits a Frobenius structure for some power of this Frobenius lift.

We then note that first assertion follows from Corollary 4.7.2 plus the off-centered Frobenius antecedent theorem (Theorem 3.3.4). Given this, we can argue that the function $r_1 \mapsto f_1(T_{[\bar{h}]}^* M, r_1, s) + \dots + f_i(T_{[\bar{h}]}^* M, r_1, s)$ is convex (Lemma 4.3.3), tends to 0 as $r_1 \rightarrow 0^+$ (Lemma 4.4.4, applicable by our first observation), and agrees with a certain linear function at the values $r_1 = r$ and $r_1 = r/q$. It must then agree with that linear function for all $r_1 \in (0, r]$, yielding the second assertion. \square

Lemma 4.7.4. *Let σ be the standard Frobenius lift. Suppose $\alpha \in \mathbb{D}_\ell$, and put $s_0 = -\log r(\alpha)$. Suppose $\rho \in [\epsilon, 1)$ satisfies*

$$0 \leq s_0 < \frac{\log p}{(-\log \rho)}.$$

For $s \in [0, s_0]$, choose $\bar{h} \in \ell$ with $\alpha_{\bar{h}, e^{-s}} \geq \alpha$, and define $|\cdot|_{\rho^\alpha, s} = |T_{[\bar{h}]}(\cdot)|_{\rho^v, s}$. Then the following hold.

- (a) *The quantity $|\cdot|_{\rho^\alpha, s}$ does not depend on the choice of \bar{h} .*
- (b) *Suppose that α is of type (iv). Then for fixed $b \in F_\rho \langle x \rangle$, for s sufficiently large, $|b|_{\rho^\alpha, s}$ is independent of s .*

Proof. Suppose first that $b \in (\mathfrak{o}_L[[x]] \otimes_{\mathfrak{o}_L} L) \cap F_\rho \langle x \rangle$. Then both (a) and (b) will follow from Lemma 4.7.1 once we check that for α of type (iv), for any given $P \in \ell \langle x \rangle$, $v_s(P)$ is constant for s sufficiently large. To show this, find $Q \in \ell[x]$ such that $v_0(P - Q) > v_{s_0}(P)$; then $v_s(P) = v_s(Q)$ for all $s \in [0, s_0]$, and $v_s(Q)$ is constant for s sufficiently large because α is of type (iv).

We now allow general $b \in F_\rho \langle x \rangle$; we may assume $b \neq 0$. To check (a), we choose $\overline{h}_1, \overline{h}_2$ with $\alpha_{\overline{h}_i, e^{-s}} \geq \alpha$ for $i = 1, 2$. We can then find $c \in L_{v, [\rho, \rho]}$ nonzero and $b' \in (\mathfrak{o}_L[[x]] \otimes_{\mathfrak{o}_L} L) \cap F_\rho \langle x \rangle$ such that

$$|b' - cb|_{\rho^v, 0} < |c|_{\rho^v} |T_{[\overline{h}_i]}(b)|_{\rho^v, s_0} \quad (i \in \{1, 2\}).$$

Recalling that $|\cdot|_{\rho^v, s}$ is a nonincreasing function of s , we deduce

$$\begin{aligned} |T_{[\overline{h}_i]}(b' - cb)|_{\rho^v, s} &\leq |T_{[\overline{h}_i]}(b' - cb)|_{\rho^v, 0} \\ &= |b' - cb|_{\rho^v, 0} \\ &< |T_{[\overline{h}_i]}(cb)|_{\rho^v, s}, \end{aligned}$$

so $|T_{[\overline{h}_i]}(b')|_{\rho^v, s} = |T_{[\overline{h}_i]}(cb)|_{\rho^v, s}$. We may thus deduce (a) from the corresponding assertion for b' .

We may now unambiguously write $|b|_{\rho^\alpha, s}$. We next verify that $|b|_{\rho^\alpha, s}$ is bounded away from 0 for $s \in [0, s_0]$. For this, we may temporarily enlarge ℓ to include an element \overline{h} such that $\alpha = \alpha_{[\overline{h}], e^{-s_0}}$; we then have $|b|_{\rho^\alpha, s} \geq |b|_{\rho^\alpha, s_0} > 0$.

We now repeat our previous enlargement of ℓ , in order to check (b). Suppose that α is of type (iv). We may then find $c \in L_{v, [\rho, \rho]}$ nonzero and $b' \in (\mathfrak{o}_L[[x]] \otimes_{\mathfrak{o}_L} L) \cap F_\rho \langle x \rangle$ such that

$$|b' - cb|_{\rho^v, 0} < |c|_{\rho^v} \inf_{s \in [0, s_0]} \{|b|_{\rho^\alpha, s}\}.$$

Then $|b' - cb|_{\rho^\alpha, s} \leq |b' - cb|_{\rho^\alpha, 0} = |b' - cb|_{\rho^v, 0} < |cb|_{\rho^\alpha, s}$, so $|b'|_{\rho^\alpha, s} = |cb|_{\rho^\alpha, s}$ and we may deduce (b) from the corresponding assertion for b' . \square

Proposition 4.7.5. *Suppose $\alpha \in \mathbb{D}_\ell$ is of type (iv), and put $s_0 = -\log r(\alpha)$. For $i = 1, \dots, n$, if $b_i(M, \alpha) > s_0$, then $b_i(M, \alpha, s)$ is constant for s in some neighborhood of s_0 .*

Proof. We proceed on induction by i . Assume that $B = b_i(M, \alpha) - s_0$ is positive, and that for $j = 1, \dots, i$, $b_j(M, \alpha, s)$ is constant for s in some neighborhood of s_0 . Note that there is no harm in replacing M by T_h^*M (since the latter carries a Frobenius structure, as in the proof of Corollary 4.7.3), or in rescaling x as long as we remain on the generic path to α ; this has the effect of reducing s_0 to some smaller value s_1 , and shifting the graph of the function $s \mapsto b_i(M, \alpha, s) - s$ to the left by $s_0 - s_1$. Using such a reparametrization, we may put ourselves in the situation where each $b_j(M, \alpha, s)$ is affine-linear (by Proposition 4.6.4); in particular, we have $b_i(M, \alpha, s) = B + s_0 + c(s_0 - s)$ for some c . We may also ensure that

$$ps_0 < b_i(M, \alpha, 0) + |c|s_0 < p(B - 2|c|s_0) \tag{4.7.5.1}$$

since we can make s_0 arbitrarily close to 0 and $b_i(M, \alpha, 0)$ arbitrarily close to B . We may then replace M by T_h^*M if needed to ensure that α does not belong to the open unit disc. We may then choose ϵ so that $f_1(M, r, 0) < \frac{p}{p-1} \log p$ for all $r \in (0, -\log \epsilon]$.

Choose $r \in (0, -\log \epsilon)$ for which there exists a nonnegative integer m with

$$\frac{q^{-m}}{p-1} \log p + 2|c|s_0 r < Br, \quad b_i(M, \alpha, 0)r + |c|s_0 r < \frac{q^{-m}p}{p-1} \log p \quad (4.7.5.2)$$

(this is possible thanks to the second inequality in (4.7.5.1)). From the second inequality in (4.7.5.2) and the first inequality in (4.7.5.1), we deduce

$$q^m r s_0 < \frac{1}{p-1} \log p. \quad (4.7.5.3)$$

For any $s \in (0, s_0)$, choose $\bar{h}(s) \in \mathfrak{o}_\ell$ with $\alpha_{\bar{h}(s), e^{-s}} \geq \alpha$; by construction, this forces $v(\bar{h}(s)) = 0$. Then by Corollary 4.7.3,

$$f_j(T_{[\bar{h}(s)]}^* M, r, s) = b_j(M, \alpha, s)r \quad (j = 1, \dots, n). \quad (4.7.5.4)$$

By Proposition 4.6.4, $b_1(M, \alpha, s) + \dots + b_i(M, \alpha, s)$ is nonincreasing for $s \in (0, s_0)$. However, we have by the induction hypothesis that $b_1(M, \alpha, s), \dots, b_{i-1}(M, \alpha, s)$ are constant (only *a priori* in a neighborhood of s_0 , but we already ensured that each $b_j(M, \alpha, s)$ is affine-linear). Hence $b_i(M, \alpha, s)$ is nonincreasing; in particular, we have $b_i(M, \alpha, s)r \leq b_i(M, \alpha, 0)r$. Combining this with (4.7.5.4) and (4.7.5.2), we deduce $f_i(T_{[\bar{h}(s)]}^* M, r, s) < \frac{q^{-m}p}{p-1} \log p$.

Let σ be the standard Frobenius lift. By Corollary 3.3.7 applied repeatedly,

$$f_{i+(q^m-1)n}(T_{[\bar{h}(s)]}^* \sigma_*^m M, q^m r, s) = q^m f_i(T_{[\bar{h}(s)]}^* M, r, s) \quad (4.7.5.5)$$

and

$$\sum_{j=1}^{i+(q^m-1)n} f_j(T_{[\bar{h}(s)]}^* \sigma_*^m M, q^m r, s) = \frac{p(q^m - 1)}{p-1} n \log p + q^m \sum_{j=1}^i f_j(T_{[\bar{h}(s)]}^* M, r, s). \quad (4.7.5.6)$$

Put $\rho = e^{-rq^m}$. Pick a cyclic vector of $\sigma_*^m M$ and define the polynomial P as in Theorem 3.4.2. By Lemma 4.7.4 (whose hypotheses are satisfied thanks to (4.7.5.3)), for each j , $|P_j|_{\rho^\alpha, s}$ becomes constant for s sufficiently close to s_0 . There is thus a terminal shape of the Newton polygon of P .

By both inequalities of (4.7.5.2),

$$\begin{aligned} \frac{q^{-m}}{p-1} \log p + rs &< Br - 2|c|rs_0 + rs \\ &\leq (B + s_0 + c(s_0 - s))r \\ &= b_i(M, \alpha, s)r \\ &= b_i(M, \alpha, 0)r - crs \\ &\leq b_i(M, \alpha, 0)r + |c|rs \\ &< \frac{q^{-m}p}{p-1} \log p + rs. \end{aligned}$$

By (4.7.5.4) and (4.7.5.5),

$$b_i(M, \alpha, s)r = f_i(T_{[\bar{h}(s)]}^* M, r, s) = q^{-m} f_{i+(q^m-1)n}(T_{[\bar{h}(s)]}^* \sigma_*^m M, q^m r, s).$$

We may thus deduce

$$f_{i+(q^m-1)n}(T_{[\bar{h}(s)]}^* \sigma_*^m M, q^m r, s) \in \left(\frac{1}{p-1} \log p + q^m rs, \frac{p}{p-1} \log p + q^m rs \right). \quad (4.7.5.7)$$

Thanks to (4.7.5.7), Theorem 3.4.2 allows us to read off the values of $f_j(T_{[\bar{h}(s)]}^* \sigma_*^m M, q^m r, s)$ for $j = 1, \dots, i+(q^m-1)n$ from the Newton polygon of P under $|\cdot|_{\rho^\alpha, s}$. By (4.7.5.6), we may deduce that $b_1(M, \alpha, s) + \dots + b_i(M, \alpha, s)$ is constant for s sufficiently large; by the induction hypothesis, this implies that $b_i(M, \alpha, s)$ is constant for s sufficiently large, as desired. \square

Remark 4.7.6. In trying to understand the proof of Proposition 4.7.5, it may help to think of B as playing the role of an absolute measure, which dictates the choice of the other auxiliary parameters r, m in order to put the situation into the range where cyclic vectors and Theorem 3.4.2 become useful. This crucial role means that we do not have a similar argument in case $B = 0$.

Corollary 4.7.7. Suppose $\alpha \in \mathbb{D}_\ell$ is minimal under domination (i.e., is of type (i) or (iv)). For $i = 1, \dots, n$, the terminal slope of $b_i(M, \alpha, s)$ is at most 1.

Proof. Put $s_0 = -\log r(\alpha)$. If α is of type (i), then all the terminal slopes are equal to 1 by Remark 4.6.6, so we assume hereafter that α is of type (iv). If $b_i(M, \alpha) > s_0$, then Proposition 4.7.5 implies that the terminal slope of $b_i(M, \alpha, s)$ is zero. Otherwise, for s slightly less than s_0 , the function $b_i(M, \alpha, s) - s$ is nonnegative and affine-linear with limit 0 as $s \rightarrow s_0^-$, so its slope must be nonpositive. Hence the terminal slope of $b_i(M, \alpha, s)$ is at most 1. \square

Remark 4.7.8. Note that we have not ruled out the possibility that the terminal slope is in the range $(0, 1)$, in the case where α is of type (iv) and $b_i(M, \alpha) = s_0$. However, in case the terminal slope is known to be an integer, it is then forced to either equal 1 (in which case $b_i(M, \alpha, s) = s$ for s in a neighborhood of s_0) or to be nonpositive.

4.8 Dwork modules

Throughout this subsection, retain Hypothesis 4.6.1. We now make a more careful study of the limiting behavior of $b_1(M, \alpha, s)$ when M is a Dwork module.

Lemma 4.8.1. For any $P_1, \dots, P_m \in \ell[x]$ and any $r_1, \dots, r_m \geq 0$, the subset

$$\{\beta \in \mathbb{D}_\ell : |P_i|_\beta \leq r_i \quad (i = 1, \dots, m)\}$$

is a (possibly empty) finite union of discs, each of which contains an ℓ -rational point. In particular, if this set is nonempty, then it contains an ℓ -rational point.

Proof. Note that the intersection of two discs D_1, D_2 is either D_1, D_2 , or the empty set. It thus suffices to check the case $i = 1$, which we treat as follows.

Factor $P_1 = c(x - z_1) \cdots (x - z_l)$. If $l = 0$, then the subset is either empty or all of \mathbb{D}_ℓ . Otherwise, we proceed by induction on l . Put $r = \max_{i \neq j} \{|z_i - z_j|\}$. If $r_1 \geq |c|r^l$, then the desired region is precisely the disc with center z_1 and radius $(r_1/|c|)^{1/l}$. (This includes the case where the z_i are all equal, as then $r = 0$.) Otherwise, the region is contained in the union of the open discs of radius r with centers z_1, \dots, z_l . The disc containing z_i fails to contain some z_j , so on that disc $|x - z_j|$ is identically equal to $|z_i - z_j|$. Hence we reduce to l separate problems involving polynomials of degree less than l , so the induction hypothesis finishes the job. \square

Proposition 4.8.2. *Suppose that α is of type (iv). Suppose that $M \in C_{v,[\epsilon,1)}$ is the free module of rank 1 generated by \mathbf{v} satisfying*

$$\nabla(\mathbf{v}) = \sum_{i \neq 0 \pmod{p}} \pi i \tilde{u}_i x^{i-1} \mathbf{v} \otimes dx$$

for some $\tilde{u}_i \in \mathfrak{o}_L$, all but finitely many of which are zero, and some $\pi \in K$ satisfying $\pi^{p-1} = -p$. Assume also that $b_1(M, \alpha, s)$ is affine-linear on $[0, s_0]$. Then the slope of $b_1(M, \alpha, s)$ equals 0 unless $b_1(M, \alpha, s) - s$ is identically zero.

Note that the existence of a Frobenius structure on M is a consequence of the fact that the series $\exp(\pi x - \pi x^q)$ converges in a disc of radius strictly greater than 1. Also note that if $b_1(M, \alpha) > s_0$, then the claim follows from Proposition 4.7.5, although we will not use this in the proof.

Proof. We may assume that $b_1(M, \alpha, s) - s$ is not identically zero, so that by Proposition 4.6.4, the slope of $b_1(M, \alpha, s)$ is some nonpositive integer. By Lemma 3.5.1,

$$b_1(M, s) = \max\{s, \max_{i \neq 0} \{-v(u_i) + (1 - i)s\}\}$$

where $u_i \in \ell$ is the reduction of \tilde{u}_i . Put $u = \sum_i u_i x^i$.

For $P(x) = \sum_{i=0}^m P_i x^i \in \ell[x]$, define

$$AS(P)(x) = \sum_{j \geq 0, j \neq 0 \pmod{p}} x^j \sum_{i=0}^{\infty} P_{jp^i}^{1/p^i}.$$

For $s \in [0, s_0)$, choose $\bar{h} \in \ell$ with $\alpha_{\bar{h}, e^{-s}} \geq \alpha$. Let $T_{\bar{h}} : \ell[x] \rightarrow \ell[x]$ denote the substitution $x \mapsto x + \bar{h}$. Let u'_i be the coefficient of x^i in $AS(T_{\bar{h}}(u))$. Then

$$b_1(M, \alpha, s) = \max\{s, \max_i \{-v(u'_i) + (1 - i)s\}\}.$$

Let $1 - i_0 \leq 0$ be the slope of $b_1(M, \alpha, s)$; then the term $i = i_0$ must dominate everywhere.

Now choose \bar{h} in a complete extension ℓ' of ℓ so that in fact $\alpha = \alpha_{\bar{h}, e^{-s_0}}$. Then

$$-v(u'_i) + (1-i)s \leq b_1(M, \alpha, s) \quad (s \in \{0, s_0\}). \quad (4.8.2.1)$$

We can choose a nonnegative integer l so that each u'_i is a polynomial in \bar{h}^{1/p^l} . That is, we may write $u'_i = \sum_j u'_{ij} \bar{h}^{j/p^l}$ with $u'_{ij} \in \ell$. Define

$$U'_i = \sum_j u'_{ij} x^j \in \ell[x].$$

We then define a subset S of \mathbb{D}_ℓ by putting

$$S = \{\beta \in \mathbb{D}_\ell : \log |U'_i|_\beta + (1-i)s \leq b_1(M, \alpha, s) \quad (s \in \{0, s_0\}; i \geq 0)\}.$$

The set S is nonempty because it contains $\alpha_0 = \alpha_{\bar{h}^{1/p^l}, 0}|_{\ell[x]}$. Thus by Lemma 4.8.1, S must be a nonempty union of discs each containing an ℓ -rational point.

Consequently, we can find $(\bar{h}')^{1/p^l} \in \ell$ in the intersection of S with the open unit disc containing α_0 . Put $\alpha' = \alpha_{\bar{h}', e^{-s_0}}$. Since \bar{h}' lies in the open unit disc containing \bar{h} , and since α is of type (iv), there is a greatest value $s_1 \in (0, s_0)$ where $\alpha_{\bar{h}', e^{-s_1}} \geq \alpha$. We now have that $b_1(M, \alpha, s) = b_1(M, \alpha', s)$ for $s \in [0, s_1]$, but $b_1(M, \alpha, s) \geq b_1(M, \alpha', s)$ for $s \in [0, s_0]$ by (4.8.2.1). Since the function $s \mapsto b_1(M, \alpha, s)$ is affine by hypothesis while the function $s \mapsto b_1(M, \alpha', s)$ is convex by Proposition 4.6.4, we conclude that $b_1(M, \alpha, s) = b_1(M, \alpha', s)$ for $s \in [0, s_0]$.

By Theorem 3.1.7 applied after a rescaling, the left slope of $b_1(M, \alpha, s)$ at $s = s_1$ must be less than or equal to the sum of the right slopes of $b_1(M, \alpha, s)$ and $b_1(M, \alpha', s)$ at $s = s_1$. Consequently, $(1 - i_0) \leq 2(1 - i_0)$, whence $1 - i_0 \geq 0$. Since $1 - i_0 \leq 0$ from earlier, the only possibility is $i_0 = 1$, so the slope of $b_1(M, \alpha, s)$ is 0, as desired. \square

4.9 Shrinking the domain of definition

We will sometimes need a variation on the construction of the category $C_{v,I}$, corresponding to working on an annulus rather than a disc in the x -direction. We no longer retain Hypothesis 4.6.1.

Definition 4.9.1. For $I \subseteq (0, 1)$ an interval and $J \subseteq [0, +\infty]$ another interval, let $R_{v,I,J}$ be the Fréchet completion of $L_{v,I}[x]$ (if $+\infty \in J$) or $L_{v,I}[x, x^{-1}]$ (if $+\infty \notin J$) for the norms $|\cdot|_{\rho^v, s}$ for $\rho \in I$, $s \in J$.

Lemma 4.9.2. Let $J \subseteq J' \subseteq [0, +\infty]$ be intervals. Suppose $f \in R_{v,I,J}$ has the property that for each $\rho \in I$, the image of f in $\Gamma(A_{F_\rho}(\rho^J), \mathcal{O})$ lifts to $\Gamma(A_{F_\rho}(\rho^{J'}), \mathcal{O})$. Then f lifts to $R_{v,I,J'}$.

Proof. In $\Gamma(A_{F_\rho}(\rho^J), \mathcal{O})$, we can write $f = \sum_{i \in \mathbb{Z}} f_i x^i$ with $f_i \in F_\rho$. In order for f to lift to $R_{v,I,J}$, we must have $f_i \in L_{v,I}$ for each i ; in order for f to lift to $\Gamma(A_{F_\rho}(\rho^{J'}), \mathcal{O})$, for each $s \in J'$ we must have $|f_i|_{\rho^v} \rho^{s_i} \rightarrow 0$ as $i \rightarrow \pm\infty$. These conditions together imply $f \in R_{v,I,J'}$. \square

Definition 4.9.3. Let $I_1 \subseteq I_2 \subseteq \dots$ be an increasing sequence of closed intervals with union I , and let $J_1 \subseteq J_2 \subseteq \dots$ be an increasing sequence of closed intervals with union J . (If $+\infty \in J$, we require $+\infty \in J_i$ for all i .) For this definition only, put $R_i = R_{v,I_i,J_i}$. Let $C_{v,I,J}$ be the category whose elements are sequences M_1, M_2, \dots with M_i a finite locally free ∇ -module over R_i , equipped with isomorphisms $M_{i+1} \otimes_{R_{i+1}} R_i \cong M_i$. Note that this category is canonically independent of the choice of the I_i and J_i , and that $C_{v,I,[0,+\infty]} = C_{v,I}$. Let $C_{v,*J}$ be the direct limit of $C_{v,[\epsilon,1],J}$ as $\epsilon \rightarrow 1^-$.

Definition 4.9.4. For $M \in C_{v,I,J}$ and $\rho \in I$, we may base extend M to obtain a ∇ -module M_ρ on the annulus $A_{F_\rho}(\rho^J)$. As in Definition 4.2.4, we call M_ρ the *cross section* of M at ρ . For $s \in I$, put $R(M, \rho, s) = R((M_\rho)_{\rho^s})$ and $f_i(M, r, s) = f_i(M_{e^{-r}}, rs)$ according to Definition 3.1.3.

Lemma 4.9.5. Let $J \subseteq J' \subseteq [0, +\infty]$ be intervals. Suppose $M \in C_{v,I,J'}$ and that \mathbf{v} is a section of the restriction of M to $C_{v,I,J}$, such that for each $\rho \in I$, the image of \mathbf{v} in M_ρ lifts from $A_{F_\rho}(\rho^J)$ to $A_{F_\rho}(\rho^{J'})$. Then \mathbf{v} is a section of M itself.

Proof. This follows from Lemma 4.9.2 using [23, Lemma 2.3.1]. \square

As in Lemma 4.3.3, we obtain the following.

Lemma 4.9.6. Take $M \in C_{v,I,J}$ of rank n . For $i = 1, \dots, n$, for $s \in J \setminus \{+\infty\}$ fixed, the function

$$r \mapsto f_1(M, r, s) + \dots + f_i(M, r, s)$$

is convex.

Definition 4.9.7. Let $\sigma : R_{v,[\epsilon,1)} \rightarrow R_{v,[\epsilon^{1/q},1)}$ be a Frobenius lift. For $M \in C_{v,[\epsilon,1),J}$, a *Frobenius structure* on M with respect to σ is an isomorphism $F : \sigma^*M \cong M$ of objects in $C_{v,[\epsilon^{1/q},1),J}$.

As in Corollary 4.5.2, we obtain the following.

Lemma 4.9.8. Suppose J is closed, and $M \in C_{v,[\epsilon,1),J}$ is of rank n and admits a Frobenius structure. Then there exist $r_0 \in (0, -\log \epsilon]$ and constants $b_i(M, s)$ for $i = 1, \dots, n$ such that for $r \in (0, r_0]$ and $s \in J \setminus \{+\infty\}$, $f_i(M, r, s) = b_i(M, s)r$ for $i = 1, \dots, n$.

As in Proposition 4.6.4, we obtain the following.

Lemma 4.9.9. For $M \in C_{v,[\epsilon,1),J}$ of rank n , the functions $b_i(M, s)$ for $i = 1, \dots, n$ have the following properties.

- (a) Each $b_i(M, s)$ is continuous and piecewise affine-linear on $J \setminus \{+\infty\}$ with slopes in $\frac{1}{n!}\mathbb{Z}$.
- (b) The function $b_1(M, s) + \dots + b_i(M, s)$ is convex. (Beware that we have no nonincreasing assertion here.)

Definition 4.9.10. Assume Hypothesis 4.6.1. Suppose $\alpha \in \mathbb{D}_\ell$ and $J \subseteq [0, -\log r(\alpha)]$. If α is of type (iv), assume further that $J \subseteq [0, s]$ for some $s < -\log r(\alpha)$. We may then choose $\bar{h} \in \mathfrak{o}_\ell$ such that $\alpha_{\bar{h},e^{-s}} \geq \alpha$ for each $s \in J$. Choose $h \in L_{v,\epsilon}$ with $|h|_L \leq 1$ lifting \bar{h} . Define the category $C_{v,*,J,h}$ to be a copy of $C_{v,*,J}$.

Choose now $\bar{h}' \in \mathfrak{o}_\ell$ such that $\alpha_{\bar{h}',e^{-s}} \geq \alpha$ for each $s \in J$, and $h' \in L_{v,\epsilon}$ with $|h'|_L \leq 1$ lifting \bar{h}' . If $\bar{h} \neq \bar{h}'$, for $\rho \in (0, 1)$ sufficiently close to 1, we have $|h - h'|_{\rho^v} = \rho^{v(\bar{h}-\bar{h}')}$. If $\bar{h} = \bar{h}'$, then as $\rho \rightarrow 1^-$, $|h - h'|_{\rho^v}$ tends to a limit strictly less than 1. For $s \in J$, we have $\alpha_{\bar{h},e^{-s}} \geq \alpha$ and $\alpha_{\bar{h}',e^{-s}} \geq \alpha$, so $\alpha_{\bar{h},e^{-s}} = \alpha_{\bar{h}',e^{-s}}$ by Lemma 2.2.12. That is, $v(\bar{h} - \bar{h}') \geq s$, so for $\rho \in (0, 1)$ sufficiently close to 1 (uniformly in s), we have $\rho^s \geq |h - h'|_{\rho^v}$. For such ρ , the norm $|\cdot|_{\rho^v,s}$ is invariant under $T_{h-h'}$ as in Definition 4.5.3, so $T_{h'-h}$ induces an endomorphism of $R_{v,[\epsilon,1],J}$ for $\epsilon \in (0, 1)$ sufficiently close to 1.

We thus obtain an isomorphism $T_{h'-h}^* : C_{v,*,J,h} \rightarrow C_{v,*,J,h'}$; since these isomorphisms satisfy the cocycle condition, we may use them to identify $C_{v,*,J,h}$ and $C_{v,*,J,h'}$ with a single category $C_{v,*,J,\alpha}$. Note that the functions $b_i(\cdot, s)$ for $s \in J$ commute with these isomorphisms thanks to Lemma 4.5.4; thus they induce well-defined functions $b_i(\cdot, \alpha, s)$ on $C_{v,*,J,\alpha}$ (in a manner consistent with Definition 4.6.3).

For $J' \subseteq J$, we have restriction functors $C_{v,*,J,\alpha} \rightarrow C_{v,*,J',\alpha}$. Using these, if α is of type (i) or (iv), we define $C_{v,*,*,\alpha}$ to be the inverse limit of $C_{v,*,J,\alpha}$ over increasing intervals $J \subseteq [0, -\log r(\alpha)]$. Again, the function $b_i(M, \alpha, s)$ is well-defined for $s \in [0, -\log r(\alpha)]$.

For $s \in [0, -\log r(\alpha)]$, define $C_{v,*,s,\alpha}$ to be the inverse limit of $C_{v,*,J,\alpha}$ over increasing intervals $J \subseteq [s, -\log r(\alpha)]$. Then define $C_{v,*,**,*,\alpha}$ to be the direct limit of $C_{v,*,s,\alpha}$ over $s \in [0, -\log r(\alpha)]$.

Remark 4.9.11. We will use later the fact that if $\rho \in I$ and $0 \in J$, then within $L'_{v,[\rho,\rho]}$,

$$L'_{v,I} \cap A_{F_\rho}(\rho^J) = R_{v,I,J}.$$

This is most easily seen by representing elements of $L'_{v,[\rho,\rho]}$ as Laurent series in x with coefficients in $L_{v,[\rho,\rho]}$.

5 Induction on transcendence defect

In this section, we carry out an induction on transcendence defect in order to prove local semistable reduction. Specifically, we prove this result, which completes the proofs of local (Theorem 2.4.3) and global (Theorem 2.4.4) semistable reduction. (See § 5.7 for the overall structure of the proof of Theorem 5.0.1.)

Theorem 5.0.1. *Assume that k is algebraically closed. Suppose that for some nonnegative integer n , local semistable reduction (Definition 2.4.1) holds whenever the valuation v is minimal of transcendence defect at most n . Then local semistable reduction also holds when v is minimal of transcendence defect $n + 1$.*

Remark 5.0.2. Note that in the course of showing Theorem 5.0.1, it will be convenient to replace K by a finite extension. This is harmless, because log-extendability of an F -isocrystal can be checked after a finite extension of the constant field, e.g., by Galois descent. (In fact, log-extendability can even be checked after an arbitrary complete extension of the constant field, but this requires a more robust argument using linear compactness.)

Notation 5.0.3. We write $\text{End}(\ast)$ as shorthand for $\ast^\vee \otimes \ast$.

5.1 Setup: relative geometry

We first do a little geometry over k in order to set up a framework, using fibrations in curves, in which we will be able to analyze a valuation of positive transcendence defect.

Hypothesis 5.1.1. Throughout this section, assume that k is algebraically closed. Fix a nonnegative integer n , and assume that local semistable reduction holds whenever the valuation v is minimal of transcendence defect at most n . Let X be a smooth irreducible k -variety, let Z be a closed subvariety of X , let v be a minimal valuation on $k(X)$ of transcendence defect $n+1$ centered on X , and let \mathcal{E} be an F -isocrystal on $X \setminus Z$ overconvergent along Z . (Note that by Abhyankar's inequality, we must have $\dim(X) \geq \text{trdefect}(v) + 1 = n+2 \geq 2$.)

To prove Theorem 5.0.1, it is enough to show that under Hypothesis 5.1.1, \mathcal{E} admits local semistable reduction at v . To do this, we first make some additional geometric arrangements.

Lemma 5.1.2. *Under Hypothesis 5.1.1, there exists a local alteration $f : \tilde{X} \rightarrow X$ around v , a closed subscheme \tilde{Z} of \tilde{X} containing $f^{-1}(Z)$, and an extension \tilde{v} of v to $k(\tilde{X})$, such that if we replace X, Z, v by $\tilde{X}, \tilde{Z}, \tilde{v}$, then Hypothesis 5.1.3 below can be satisfied.*

Before proving this, we introduce the geometric hypothesis we are trying to fulfill, and make some remarks about it.

Hypothesis 5.1.3. For the remainder of this section (except the proof of Lemma 5.1.2, and Remark 5.1.7), suppose that there exist $X^0, Z^0, \pi, v^0, z^0, z, x$ as follows.

- The k -variety X^0 is smooth irreducible affine, and Z^0 is a closed subvariety of X^0 .
- The morphism $\pi : X \rightarrow X^0$ is smooth of relative dimension 1 with geometrically integral generic fibre, and satisfies $Z = \pi^{-1}(Z^0)$.
- The restriction v^0 of v to $k(X^0)$ has transcendence defect n and is exposed by the pair (X^0, Z^0) . Note that this implies that $\text{ratrank}(v) = \text{ratrank}(v^0)$, and that v is exposed by (X, Z) : we obtain a good system of parameters by pulling back a good system of parameters at the center of v^0 , then adding one more parameter to complete a system of local coordinates.
- The point $z^0 \in X^0$ is the center of v^0 , and the point $z \in X$ is the center of v .

- The fibre $\pi^{-1}(z^0)$ is an affine line, and $x \in \Gamma(X, \mathcal{O})$ restricts to a coordinate of that line.

Remark 5.1.4. Under Hypothesis 5.1.3, $k(X)$ is a finite extension of $k(X^0)(x)$. Moreover, the local ring $\mathcal{O}_{X,z}$ is contained in the completion of $k(X^0)[x]$ for the 0-Gauss valuation with respect to v^0 . Consequently, $k(X)$ is contained in the completion of $k(X^0)(x)$ under any extension of v^0 which on $k(X^0)[x]$ is greater than or equal to the Gauss valuation. In particular, v is uniquely determined by its restriction to $k(X^0)(x)$.

Definition 5.1.5. Let P^0 be a smooth irreducible affine formal scheme over $\mathrm{Spf} \mathfrak{o}_K$ with $P_k^0 \cong X^0$. Let σ^0 be a q -power Frobenius lift on P^0 . We may restrict \mathcal{E} to an (F, ∇) -module on a space of the form $V \times A_{K,x}[0, 1]$, where V is the intersection of $]z^0[_{P_K^0}$ with a strict neighborhood of $]X^0 \setminus Z^0[_{P^0}$ in P_K^0 .

Once Hypothesis 5.1.3 has been enforced, we will proceed to modify the geometric situation while preserving Hypothesis 5.1.3. Here are the operations we will use to accomplish this.

Definition 5.1.6. By a *base change on Hypothesis 5.1.3*, we will mean an operation consisting of the following steps.

- Replace X^0 by \tilde{X}^0 for some local alteration $f : \tilde{X}^0 \rightarrow X^0$ around v^0 , and replace v^0 by an extension \tilde{v}^0 to $k(\tilde{X}^0)$, such that if we put $\tilde{Z}^0 = f^{-1}(Z^0)$, then \tilde{v}^0 is exposed by $(\tilde{X}^0, \tilde{Z}^0)$.
- Replace X by the Zariski closure \tilde{X} of the graph of the rational map $X \times_{X^0} \tilde{X}^0 \dashrightarrow \mathbb{A}_k^1$ given by $(x - h)/g$ for some $g \in \Gamma(\tilde{X}^0, \mathcal{O}) \cap \Gamma(\tilde{X}^0 \setminus \tilde{Z}^0, \mathcal{O}^*)$ and $h \in \Gamma(\tilde{X}^0, \mathcal{O})$ such that for some extension \tilde{v} of v to $k(\tilde{X})$ extending v on $k(X)$ and \tilde{v}^0 on $k(\tilde{X}^0)$, we have $\tilde{v}(x - h) \geq \tilde{v}(g)$. Let $\tilde{x} = (x - h)/g \in \Gamma(\tilde{X}, \mathcal{O})$.

In case $g = 1$, we call the operation a *scale-preserving base change*. In case $h = 0$, we call the operation a *center-preserving base change*. If both of these hold (so the only effect is to change X^0), we call the operation a *simple base change*. (See Remark 5.2.4 for details on how we choose the extended valuations.)

Remark 5.1.7. Note that it also makes sense to carry out the base change described in Definition 5.1.6 even if the fibre $\pi^{-1}(z_0)$ is not an affine line, as long as x restricts to a local parameter for this fibre at z . As long as $0 < \tilde{v}^0(g) \leq \tilde{v}(x - h)$, the result will then satisfy Hypothesis 5.1.3. (Such g exists because Hypothesis 5.1.1 forces $\dim(X) \geq 2$ and hence $\dim(X^0) \geq 1$.)

Definition 5.1.8. By a *tame alteration on Hypothesis 5.1.3*, we will mean an operation consisting of the following steps.

- Perform a simple base change after which the zero locus of x on X is smooth.

- Replace X by the fibre product \tilde{X} of the map $x : X \rightarrow \mathbb{A}_k^1$ with the m -th power map on \mathbb{A}_k^1 , for some positive integer m coprime to p . Let \tilde{x} be the coordinate on the source of the m -th power map, so that we have the equality $x = \tilde{x}^m$ of maps $\tilde{X} \rightarrow \mathbb{A}^1$. Then shrink if necessary to get something smooth on which an extension of v is centered.

Definition 5.1.9. By an *Artin-Schreier alteration* on Hypothesis 5.1.3, we will mean an operation consisting of the following steps.

- Perform a simple base change after which the zero locus of x on X is smooth.
- Replace X by the finite flat cover \tilde{X} with structure sheaf

$$\mathcal{O}[\tilde{x}] / (\tilde{x}^p - g^{p-1}\tilde{x} - x - a_2x^2 - \cdots - a_kx^k)$$

for some $g, a_2, \dots, a_k \in \Gamma(X^0, \mathcal{O})$ with a_2, \dots, a_k in the maximal ideal of \mathcal{O}_{X^0, z^0} . Then shrink if necessary to get something smooth on which an extension of v is centered.

To conclude the geometric discussion, we show that Hypothesis 5.1.3 can be realized, by proving Lemma 5.1.2.

Proof of Lemma 5.1.2. By Theorem 2.3.9, we may reduce to the case where (X, Z) is a smooth pair. By Lemma 2.3.5 (at the possible expense of enlarging Z), we may assume that v is exposed by (X, Z) . Choose local parameters t_1, \dots, t_r for the components of Z passing through the center z of v . Then extend this to a sequence of local parameters $t_1, \dots, t_r, x_1, \dots, x_n, x$ for X at z . Use these to construct a map $X \rightarrow \mathbb{A}_k^{r+n+1}$; by shrinking X , we can ensure that this map is étale.

By dropping the last coordinate, we get a smooth morphism $\pi : X \rightarrow X^0 = \mathbb{A}_k^{r+n}$ of relative dimension 1. Let Z^0 be the zero locus of $t_1 \cdots t_r$ on X^0 . Let v^0 be the restriction of v to $k(X^0)$, whose center z^0 lies on Z^0 .

The morphism π may not have geometrically integral generic fibre. We fix this by replacing X^0 with an exposing alteration X^1 for v^0 , and X with a connected component of $X \times_{X^0} X^1$.

We have now achieved Hypothesis 5.1.3 except that the fibre of $X \rightarrow X^0$ over z^0 is not yet an affine line. However, as noted in Remark 5.1.7, we may perform a base change in the sense of Definition 5.1.6 to rectify this. \square

5.2 Setup: differential modules

We now set up the differential modules we will use to carry out the induction on transcendence defect, and sketch the proof of Theorem 5.0.1.

Definition 5.2.1. We now enter the notational régime of § 4. Let L_0 be the completion of the direct system

$$\mathrm{Frac} \Gamma(P_K^0, \mathcal{O}) \xrightarrow{\sigma^0} \mathrm{Frac} \Gamma(P_K^0, \mathcal{O}) \xrightarrow{\sigma^0} \cdots,$$

and equip the residue field $\ell_0 = k(X^0)^{\text{perf}}$ of L_0 with the unique valuation extending v^0 . Choose an extension of v^0 to $k(X^0)^{\text{alg}}$ (again denoted v^0); as in Remark 2.1.8, this extension preserves height, rational rank, residual transcendence degree, and transcendence defect. Let ℓ be the completion of $k(X^0)^{\text{alg}}$ under v^0 (which is again algebraically closed). Put $L = L_0 \otimes_{W(\ell_0)} W(\ell)$.

We then obtain from \mathcal{E} an element $M \in C_{v^0, [\epsilon, 1]}$ for some $\epsilon \in (0, 1)$, admitting a Frobenius structure for the standard Frobenius lift σ . More precisely, with notation as in Definition 5.1.5, we have a continuous homomorphism $\Gamma(V \times A_{K,x}[0, 1], \mathcal{O}) \rightarrow R_{v^0, [\epsilon, 1]}$ for some $\epsilon \in (0, 1)$. The module M is the base change of \mathcal{E} along this homomorphism.

By Lemma 2.2.3, we can find a multiplicative seminorm α on $\ell[x]$ which agrees with $e^{-v^0(\cdot)}$ on ℓ and with $e^{-v(\cdot)}$ on $k(X^0)[x]$. Then α corresponds to a point of \mathbb{D}_ℓ , so in particular we may extend α to the completion $\ell\langle x \rangle$ of $\ell[x]$ for the 0-Gauss valuation with respect to v^0 . Let $h(\alpha)$ denote the completion of $\text{Frac}(\ell\langle x \rangle / \ker(\alpha))$ for the valuation induced by α ; note that $h(\alpha)$ contains $\ell(x)$ unless α is of type (i). As in Remark 5.1.4, the local ring $\mathcal{O}_{X,z}$ is contained in $\ell\langle x \rangle$; therefore, at least when α is not of type (i), $k(X)$ is a subfield of $h(\alpha)$, and the induced valuation is precisely v . The same turns out to be true when α is of type (i); see Remark 5.2.4.

Lemma 5.2.2. *In Definition 5.2.1, the point $\alpha \in \mathbb{D}_\ell$ is of type (i) or (iv).*

Proof. Suppose to the contrary that α is of type (ii) or (iii). Then $h(\alpha)$ is the completion of an algebraic extension of $k(X)$ (namely the compositum with ℓ over k). By Remark 2.1.8, the valuation on $h(\alpha)$ has the same rational rank and residual transcendence degree as v . However, by Lemma 2.2.18, this implies that the sum of the rational rank and residual transcendence degree on ℓ is one less than on $h(\alpha)$, implying $\text{trdefect}(v^0) = \text{trdefect}(v)$. This contradicts Hypothesis 5.1.3. \square

Definition 5.2.3. For $s_0 = -\log r(\alpha)$, define $b_i(M, \alpha, s)$ for $s \in [0, s_0]$. As observed in Definition 4.6.5, the function $b_i(M, \alpha, s)$ is affine-linear on some neighborhood of s_0 . We call the slope of $b_i(M, \alpha, s)$ in this range the *terminal slope* of $b_i(M, \alpha, s)$.

Remark 5.2.4. From now on, when we perform a base change on Hypothesis 5.1.3, we will need to maintain compatibility with Definition 5.2.1. (We also need compatibility with tame and Artin-Schreier alterations, but that is straightforward.)

To arrange the compatibility, in Definition 5.1.6, we will always identify $k(\tilde{X}^0)$ within $k(X^0)^{\text{alg}}$ so that \tilde{v}^0 agrees with the restriction of our chosen extension of v^0 . We then note that the map $\ell\langle x \rangle \rightarrow h(\alpha)$ cannot fail to be injective on $k(\tilde{X}^0)[x]$: if $P(x)$ were a nonzero element of the kernel, then we could find a polynomial $Q(x) \in k(X^0)[x]$ which becomes divisible by $P(x)$ in $k(\tilde{X}^0)[x]$, and $Q(x)$ would also be in the kernel. However, the map must be injective on $k(X^0)[x]$ since α restricts to $e^{-v(\cdot)}$ there. (Note that this argument implies that $k(X)$ is a subfield of $h(\alpha)$ even when α is of type (i), as noted in Definition 5.2.1.)

Since π has geometrically integral generic fibre, we may identify

$$k(\tilde{X}) = k(X \times_{X^0} \tilde{X}^0) = k(X) \otimes_{k(X^0)} k(\tilde{X}^0) = k(X) \otimes_{k(X^0)(x)} k(\tilde{X}^0)(x)$$

with a subfield of $h(\alpha)$, and then define the extension \tilde{v} of v by restriction from $h(\alpha)$.

If the base change is not scale-preserving, then it translates the domains of the functions $b_i(M, \alpha, s) - s$ to the left by the amount $\tilde{v}(g)$.

5.3 Artin-Schreier characters and Dwork modules

In order to make further progress, we need to collect some information about Artin-Schreier characters and their associated differential modules.

Definition 5.3.1. Let R be a connected \mathbb{F}_p -algebra, and fix a geometric point \bar{x} of $\text{Spec } R$. An *Artin-Schreier character* of R is a homomorphism

$$\tau : \pi_1(\text{Spec } R, \bar{x}) \rightarrow \mathbb{Z}/p\mathbb{Z}$$

which is discrete (i.e., has open kernel). By Artin-Schreier theory (see, e.g., [11, 1.4.5]), the group of Artin-Schreier characters is isomorphic to the cokernel of the map $\phi - 1 : R \rightarrow R$, where $\phi : R \rightarrow R$ is the p -power Frobenius. Explicitly, given $u \in R$, the extension $R[z]/(z^p - z - u)$ carries the structure of a $\mathbb{Z}/p\mathbb{Z}$ -torsor by the formula

$$g(z) = z + g \quad (g \in \mathbb{Z}/p\mathbb{Z}).$$

Definition 5.3.2. For $J \subseteq [0, +\infty]$ a closed subinterval not containing $+\infty$ (resp. containing $+\infty$), let R_J be the Fréchet completion of $\ell[x, x^{-1}]$ (resp. $\ell[x]$) for the s -Gauss valuations v_s for $s \in J$. In particular, if $J = [0, +\infty]$, we have $R_J = \ell\langle x \rangle$.

We say that $u = \sum_{i \in \mathbb{Z}} u_i x^i \in R_J$ is *prepared* if $u \in k(X^0)[x, x^{-1}]$ and $u_i = 0$ whenever i is divisible by p . Note that prepared elements form a subgroup of the additive group of R , and that a prepared element u defines the trivial Artin-Schreier character of R if and only if $v_s(u) > 0$ for all $s \in J$. (Beware however that preparedness is stable only under *center-preserving* base changes on Hypothesis 5.1.3.)

Lemma 5.3.3. Given $u \in R_J$, after a simple base change on Hypothesis 5.1.3, we can find a prepared element $u' \in R_J$ defining the same Artin-Schreier character as u .

Proof. Given $u = \sum_{i \in \mathbb{Z}} u_i x^i \in R_J$, we can make certain changes to u without changing its image in $R_J/((\phi - 1)R_J)$.

- We may replace u by $u + y$ for any $y \in R_J$ with $v_s(y) > 0$ for all $s \in J$, because the series $z = y + y^p + y^{p^2} + \dots$ converges and satisfies $z - z^p = y$. Hence we can replace u by an element of $k(X^0)^{\text{alg}}[x, x^{-1}]$; after a simple base change on Hypothesis 5.1.3, we can force u into $k(X^0)[x, x^{-1}]$.
- We may eliminate all terms with i nonzero and divisible by p .
- After a simple base change on Hypothesis 5.1.3, we may also eliminate the term $u_0 x^0$.

The resulting Artin-Schreier parameter is prepared. \square

Definition 5.3.4. Let K^σ be the fixed field of K under σ_K . Suppose that $\mathbb{F}_q \subseteq k$ and that K^σ contains a primitive p -th root of unity ζ_p . Let π be the unique solution of $\pi^{p-1} = -p$ satisfying

$$\pi \equiv \zeta_p - 1 \pmod{(\zeta_p - 1)^2}.$$

Let $J \subseteq [0, +\infty]$ be a closed subinterval, and suppose $u = \sum_{i \in \mathbb{Z}} u_i x^i \in R_J$ is prepared. Choose a lift $\tilde{u}_i \in \mathfrak{o}_L$ of u_i for each i not divisible by p , such that all but finitely many of the \tilde{u}_i are zero. Let $M_u \in C_{v^0, *, J}$ be the free (Dwork) module generated by a single generator \mathbf{v} , on which ∇ acts via

$$\nabla(\mathbf{v}) = - \sum_{i \in \mathbb{Z}} \pi i \tilde{u}_i x^{i-1} \mathbf{v} \otimes dx.$$

If u corresponds to an Artin-Schreier character τ of R_J , we also write M_τ for M_u .

Lemma 5.3.5. Set notation as in Definition 5.3.4.

- (a) The isomorphism class of M_u depends only on τ , not on the choices of u or $\tilde{u} = \sum_{i \in \mathbb{Z}} \tilde{u}_i x^i$.
- (b) For some $\epsilon \in (0, 1)$, if we extend d to

$$S = R_{v^0, [\epsilon, 1], J}[z]/((1 + \pi z)^p - (1 - p\pi\tilde{u})),$$

then regard S as an object in $C_{v^0, *, J}$ carrying an action of $\mathbb{Z}/p\mathbb{Z}$ for which $g \in \mathbb{Z}/p\mathbb{Z}$ carries $1 + \pi z$ to $\zeta_p^g(1 + \pi z)$, the eigenspace of S for the character $g \mapsto \zeta_p^g$ is isomorphic to M_u .

- (c) There exists a Frobenius structure on M_u .

- (d) We have

$$b_1(M_u, s) = \max\{s, \max_{i \neq 0} \{-v^0(u_i) + (1 - i)s\}\} \quad (s \in J).$$

Proof. Before proceeding, we observe (as in the proof of Lemma 4.5.4) that if $u_i \neq 0$, then $-\log |\tilde{u}_i|_\rho = v^0(u_i)(-\log \rho)$ for ρ sufficiently close to 1. If on the other hand $u_i = 0$, then $-\log |\tilde{u}_i|_\rho$ is bounded below by a positive constant for ρ sufficiently close to 1.

To prove (a), note that if $u, u' \in R_J$ are prepared and define the same Artin-Schreier character τ , then $u - u'$ defines the trivial Artin-Schreier character. It thus suffices to check that if τ is trivial, then M_u is trivial for any choices of the \tilde{u}_i .

If τ is trivial, then $v^0(u_i) + is > 0$ for all i and all $s \in J$. For each i for which $u_i \neq 0$, for ρ sufficiently close to 1, $|\tilde{u}_i x^i|_{\rho^{v^0}, s} = \rho^{v^0(u_i)+is}$ is bounded above by a constant less than 1 uniformly for all $s \in J$. For each i for which $u_i = 0$, $|x^i|_{\rho^{v^0}, s} = \rho^{is}$ tends to 1 uniformly in s as ρ tends to 1. Consequently, for ρ sufficiently close to 1, $|\tilde{u}_i x^i|_{\rho^{v^0}, s}$ is also bounded above by a constant less than 1 uniformly for all $s \in J$. (In both cases, the conclusion is still valid even if $+\infty \in J$, because in that case we only consider $i > 0$.) In all cases, we may deduce (as in the proof of the last assertion of Lemma 3.5.1) that the exponential $\exp(\pi \sum_i \tilde{u}_i x^i)$

converges in $R_{v^0, [\epsilon, 1], J}$ for ϵ sufficiently close to 1, and hence M_u is trivial as an element of $C_{v^0, *, J}$.

To prove (b), note that the action of d on S satisfies

$$\frac{d(1 + \pi z)}{1 + \pi z} = \frac{1}{p} \frac{d(1 - p\pi \tilde{u})}{1 - p\pi \tilde{u}} = -\frac{\pi d\tilde{u}}{1 - p\pi \tilde{u}}.$$

If we can find $y \in R_{v^0, [\epsilon, 1], J}$ such that

$$\frac{dy}{y} = \frac{p\pi^2 \tilde{u} d\tilde{u}}{1 - p\pi \tilde{u}} = -\pi d\tilde{u} + \frac{\pi d\tilde{u}}{1 - p\pi \tilde{u}},$$

we may then conclude that $y(1 + \pi z)$ is a generator for the eigenspace of S for the character $g \mapsto \zeta_p^g$ which maps to \mathbf{v} under an isomorphism with M_u . We would like to take $y = \exp(w)$ for w formally defined as

$$w = \int \frac{p\pi^2 \tilde{u} d\tilde{u}}{1 - p\pi \tilde{u}} = \int \sum_{i=0}^{\infty} p^{i+1} \pi^{i+2} \tilde{u}^{i+1} d\tilde{u} = \sum_{i=0}^{\infty} \frac{p^{i+1} \pi^{i+2}}{i+2} \tilde{u}^{i+2}.$$

This becomes valid once we observe that $|p^{i+1} \pi^{i+2}| \leq |\pi^2(i+2)|$ for each $i \geq 0$, as then we may deduce that for ρ sufficiently close to 1, $|w|_{\rho v^0, s}$ converges to a limit less than $|\pi|$ uniformly for $s \in J$.

Assertion (c) holds by the discussion following the statement of Proposition 4.8.2. It also follows from (b), since the ring S is finite étale over $R_{v^0, [\epsilon, 1], J}$ and so admits an extension of any Frobenius lift (in a manner compatible with d).

To prove (d), we again use the initial observation and then apply Lemma 3.5.1. \square

Corollary 5.3.6. *With notation as in Definition 5.3.4, suppose that $b_1(M_u, s) = s$ for $s \in J$. Let τ be the Artin-Schreier character defined by u . Then for any closed subinterval J' of the interior of J , the restriction of τ to $\pi_1(\mathrm{Spec} R_{J'}, \bar{x})$ is trivial.*

Proof. Lemma 5.3.5 implies that for each $i \neq 0$, $-v^0(u_i) + (1-i)s \leq s$ for all $s \in J$. Since the difference between the two sides is affine-linear in s with nonzero slope, the equality must be strict for all $s \in J'$. Hence as a parameter for an Artin-Schreier extension of $R_{J'}$, we may replace u by 0 without changing the extension; this proves the claim. \square

We can use this computation to classify certain Dwork subobjects of the module M defined in Definition 5.2.1. (For the next statement, recall that α is a multiplicative seminorm on $\ell[x]$ which agrees with $e^{-v^0(\cdot)}$ on ℓ and with $e^{-v(\cdot)}$ on $k(X^0)[x]$, viewed as a point of \mathbb{D}_ℓ , and that $s_0 = -\log r(\alpha)$.)

Proposition 5.3.7. *Suppose N is a subquotient of M in $C_{v^0, *, *, \alpha}$ of rank 1. Let $J \subseteq [0, s_0]$ be a closed subinterval with nonempty interior, and suppose that $\alpha_{0,e^{-s}} \geq \alpha$ for all $s \in J$. Let $u \in R_J$ be prepared, and suppose that $N \cong M_u$ in $C_{v^0, *, J, \alpha} = C_{v^0, *, J}$. Let J' be any closed subinterval of the interior of J . Then the following conditions hold.*

- (a) The function $b_1(N, \alpha, s)$ is continuous, piecewise affine-linear, and convex on $[0, s_0]$.
- (b) There exists a prepared $u_+ \in \ell\langle x \rangle$ such that $M_u \cong M_{u_+}$ in $C_{v^0, *, J', \alpha}$.
- (c) Suppose that $0 \in J$ and $M_u \cong M_{u_+}$ in $C_{v^0, *, J, \alpha}$. Then $N \cong M_{u_+}$ in $C_{v^0, *, *, \alpha}$.
- (d) The terminal slope of $b_1(N, \alpha, s)$ is equal to 0 unless $b_1(N, \alpha, s) = s$ identically in a neighborhood of s_0 .

Proof. We first note that $b_1(N, \alpha, s)$ is continuous, convex, and piecewise affine-linear on $[0, s_0]$ by Lemma 4.9.9. By Proposition 4.6.4 applied to M , for s in a neighborhood of s_0 , there is an index i such that $b_1(N, \alpha, s) = b_i(M, \alpha, s)$. In particular, $b_1(N, \alpha, s)$ is affine-linear in a neighborhood of s_0 ; this proves (a).

Put $u = \sum_{i \in \mathbb{Z}} u_i x^i$ and put $u_+ = \sum_{i > 0} u_i x^i$. Since M_{u_+} belongs to $C_{v^0, *}$, we may apply the argument of the first paragraph to $M \otimes M_{u_+}^\vee$ to deduce that for $P = N \otimes M_{u_+}^\vee$, $b_1(P, \alpha, s)$ is continuous, convex, and piecewise affine-linear on $[0, s_0]$. By Corollary 4.7.7 applied to $M \otimes M_{u_+}^\vee$, the terminal slope of $b_1(P, \alpha, s)$ is at most 1. Hence for $s \in J$, the slopes of $b_1(P, \alpha, s) = b_1(P, s)$ are all less than or equal to 1. But by Lemma 5.3.5, we have

$$b_1(P, s) = \max\{s, \max_{i < 0} \{-v^0(u_i) + (1 - i)s\}\} \quad (s \in J),$$

which can only have slopes less than or equal to 1 if $v_s(u_i x^i) \geq 0$ for all $s \in J$. As in Corollary 5.3.6, this implies that $u - u_+$ defines the trivial Artin-Schreier character of $\pi_1(\text{Spec } R_{J'}, \bar{x})$; then (b) follows as in Lemma 5.3.5(a).

Suppose that $0 \in J$ and $M_u \cong M_{u_+}$ in $C_{v^0, *, J, \alpha}$. Then P is trivial in $C_{v^0, *, J, \alpha}$, so $b_1(P, \alpha, s)$ is identically equal to s for $s \in J$. However, as in the previous paragraph, this function is convex, and in a neighborhood of s_0 is affine-linear with slope at most 1. Hence we must have $b_1(P, \alpha, s) = s$ identically for $s \in [0, s_0]$. To check that P is trivial in $C_{v^0, *, *, \alpha}$, it suffices to check the triviality in $C_{v^0, *, [0, s_1], \alpha}$ for each $s_1 \in [0, s_0]$. We may check this, at the expense of losing the preparedness of u and u_+ (for the remainder of this paragraph), by applying a scale-preserving base change on Hypothesis 5.1.3 to reduce to the case where $\alpha_{0, e^{-s_1}} \geq \alpha$, so $C_{v^0, *, [0, s_1]} = C_{v^0, *, [0, s_1], \alpha}$. We then have a basis $\mathbf{v}_1, \dots, \mathbf{v}_d$ of horizontal sections of P in $C_{v^0, *, J}$, such that for each ρ sufficiently close to 1, the restriction of each \mathbf{v}_i to P_ρ extends from $A_{F_\rho}(\rho^J)$ to $A_{F_\rho}[\rho^{s_1}, 1]$ (by Theorem 3.2.4). By Lemma 4.9.5, each \mathbf{v}_i is a horizontal section of P in $C_{v^0, *, [0, s_1], \alpha}$. As noted above, this suffices to prove (c).

To prove (d), note that by (b), there exists a prepared $u_+ \in \ell\langle x \rangle$ such that $M_u \cong M_{u_+}$ in $C_{v^0, *, J', \alpha}$. Since it suffices to check (d) after a base change on Hypothesis 5.1.3, we can force the same conclusion to hold, but with J' replaced by an interval containing 0. By (c), we have $N \cong M_{u_+}$ in $C_{v^0, *, *, \alpha}$, so to check (d) it now suffices to check its analogue with N replaced by M_{u_+} .

If α is of type (i), then $b_1(M_{u_+}, \alpha, s) = s$ identically in a neighborhood of s_0 . If α is of type (iv), Proposition 4.8.2 (applied after a base change on Hypothesis 5.1.3) implies that either $b_1(M_{u_+}, \alpha, s)$ has terminal slope 0, or $b_1(M_{u_+}, \alpha, s) = s$ identically in a neighborhood of s_0 . This proves (d). \square

5.4 Monodromy representations

Having given a functor from certain representations to differential modules, we now go in the reverse direction. This step is where we first take input from the induction hypothesis, by invoking local semistable reduction for valuations of transcendence defect n .

Hypothesis 5.4.1. Throughout this subsection, let $s_1 > 0$ be a real number not in the divisible closure of the value group Γ_{v^0} of v^0 . Note that $\Gamma_{v^0} \neq \mathbb{R}$ because Γ_{v^0} has rational rank at most $\dim(X)$ by Abhyankar's inequality, so it is possible to choose such an s_1 . Write v_1 as shorthand for v_{s_1} , as a valuation on $\ell(x)$. By Remark 5.1.4, we may view $k(X)$ as a subfield of the completion $\ell(x)_{v_1}$, and thus restrict v_1 to $k(X)$.

Definition 5.4.2. Let $I_1 = \pi_1(\mathrm{Spec} k(X)_{v_1}, \bar{x})$ denote the inertia group of $\pi_1(X, \bar{x})$ for the valuation v_1 , for \bar{x} some geometric point of $\mathrm{Spec} \ell(x)_{v_1}$. By Lemma 2.2.18, v_1 has transcendence defect n , so by Hypothesis 5.1.1, we are granted local semistable reduction at v_1 . We may thus let $\tau : I_1 \rightarrow \mathrm{GL}(V)$ denote the semisimplified local monodromy representation of \mathcal{E} at v_1 , as in Definition 2.5.14.

Remark 5.4.3. Retain notation as in Definition 5.4.2. We can gain some information about τ by using the subgroup structure on I_1 coming from Definition 2.5.6. First, denote $I'_1 = \pi_1(\mathrm{Spec} \ell(x)_{v_1}, \bar{x})$ viewed as a subgroup of I_1 . The subgroup $\tau^{-1}(\tau(I'_1)) \subseteq I_1$ is open, corresponding to the étale fundamental group of a finite separable extension of $k(X)_{v_1}$ contained in $\ell(x)_{v_1}$. Apply the primitive element theorem to write this extension as $k(X)_{v_1}(\gamma)$ with $\gamma \in \ell(x)_{v_1}$, and let $P(T)$ be the minimal polynomial of γ . Since $k(X^0)^{\mathrm{alg}}(x)$ is dense in $\ell(x)_{v_1}$, we can choose $\delta \in k(X^0)^{\mathrm{alg}}(x)$ such that $v_1(\gamma - \delta) > v_1(\gamma' - \gamma)$ for each root $\gamma' \neq \gamma$ of $P(T)$. After a suitable simple base change on Hypothesis 5.1.3, we can force $\delta \in k(X^0)(x)$. Then Hensel's lemma implies that $\gamma \in k(X)_{v_1}$, in which case $\tau(I_1) = \tau(I'_1)$.

Second, assuming $\tau(I_1) = \tau(I'_1)$, let W'_1 denote the wild inertia subgroup of I'_1 . The subgroup $\tau^{-1}(\tau(W'_1)) \subseteq I_1$ is open, corresponding to the étale fundamental group of a finite *tamely ramified* extension of $k(X)_{v_1}$. Thus after a suitable simple base change and tame alteration on Hypothesis 5.1.3, we have $\tau(I_1) = \tau(W'_1)$. In this case, $\tau(I_1)$ becomes a p -group since W'_1 is a pro- p -group.

Lemma 5.4.4. *After performing a suitable simple base change on Hypothesis 5.1.3, there exist:*

- *some $t_1, \dots, t_r \in \Gamma(P^0, \mathcal{O})$ whose images in $\Gamma(P_k^0, \mathcal{O})$ cut out the components of Z^0 passing through z^0 ;*
- *for $i = 1, \dots, r$, some $a_i, b_i \in \mathbb{Z}_{(p)}$ with $a_i < v^0(t_i)/s_1 < b_i$;*
- *some $\epsilon \in (0, 1)$;*

such that when applying Definition 2.5.14 to v_1 , we can take the localizing subspace A to have the form

$$\{y \in z^0[P^0 \times A_{K,x}[0, 1]] : |x(y)| \in (\epsilon, 1), \quad |t_i(y)| \in (|x(y)|^{b_i}, |x(y)|^{a_i}) \quad (i = 1, \dots, r)\}.$$

Proof. We first verify that if we start with a subspace of the desired form, then apply a simple base change on Hypothesis 5.1.3, the inverse image of the original subspace contains another subspace of the desired form. Namely, if \tilde{P}_0 is a smooth irreducible affine formal scheme over $\mathrm{Spf} \mathfrak{o}_K$ with special fibre \tilde{X}^0 , and $\tilde{t}_1, \dots, \tilde{t}_r \in \Gamma(\tilde{P}^0, \mathcal{O})$ have images in $\Gamma(\tilde{P}_k^0, \mathcal{O})$ cutting out the components of \tilde{Z}^0 passing through \tilde{z}^0 , then there is a unique invertible $r \times r$ matrix A over \mathbb{Q} such that

$$\tilde{v}^0(\tilde{t}_j) = \sum_{i=1}^r A_{ij} v^0(t_i).$$

Since s_1 is not in the \mathbb{Q} -span of Γ_{v^0} , the quantity $v^0(t_i)/s_1$ must belong not just to the closed interval $[a_i, b_i]$ but also to the open interval (a_i, b_i) . Applying the linear transformation on column vectors defined by A to the product $\prod_{i=1}^r (a_i, b_i) \subseteq \mathbb{R}^r$ gives an open subset of \mathbb{R}^r . Inside that open subset, we can find a product $\prod_{i=1}^r (a'_i, b'_i)$ with $a'_i, b'_i \in \mathbb{Z}_{(p)}$ and $a'_i < \tilde{v}_0(t'_i)/s_1 < b'_i$. Then the subset

$$\{y \in \tilde{P}_k^0 \times A_{K,x}[0, 1] : |x(y)| \in (\epsilon, 1), \quad |\tilde{t}_i(y)| \in (|x(y)|^{b'_i}, |x(y)|^{a'_i}) \quad (i = 1, \dots, r)\}$$

is contained in the inverse image of the original subspace.

Before continuing, we note that given any $e_1, \dots, e_r \in \mathbb{Q}$ and $j \in \mathbb{Z}$ such that $v_1(t_1^{e_1} \cdots t_r^{e_r} x^j) > 0$, we can adjust the choice of the a_i and b_i (without any base change on Hypothesis 5.1.3) to ensure that $|t_1^{e_1} \cdots t_r^{e_r} x^j| < 1$ everywhere on the localizing subspace. This follows by the argument using linear transformations from the previous paragraph.

It remains to verify that we can shrink the localizing subspace so as to satisfy an additional condition of the form in the definition of a localizing subspace. We split this verification into two steps. We first consider a condition of the form $\delta \leq |f| < 1$ for f (defined on some smooth formal lift of X) having reduction $\bar{f} \in \Gamma(X, \mathcal{O})$ with $v_1(\bar{f}) > 0$. Expand $\bar{f} = \sum_{i=0}^{\infty} \bar{f}_i x^i$ in $\mathcal{O}_{X^0, z^0}[[x]]$. Since s_1 is positive and not in Γ_{v^0} , there must exist an index h such that $v^0(\bar{f}_i) + is_1 > v^0(\bar{f}_h) + hs_1$ for all $i \neq h$; there must also exist an index $N > h$ such that $(N - h)s_1 > v^0(\bar{f}_h)$.

By performing a suitable simple base change on Hypothesis 5.1.3, then adjusting a_i and b_i as above, we can ensure that the following conditions hold.

- (a) For $i = 0, \dots, N - 1$, \bar{f}_i can be written as the product of a unit $\bar{u}_i \in \Gamma(X^0, \mathcal{O})^\times$ with the monomial $t_1^{e_{i1}} \cdots t_r^{e_{ir}}$ for some nonnegative integers e_{i1}, \dots, e_{ir} .
- (b) For $i = 0, \dots, N - 1$ with $i \neq h$, we have $|t_1^{e_{i1}-e_{h1}} \cdots t_r^{e_{ir}-e_{hr}} x^{i-h}| < 1$ everywhere on the localizing subspace.
- (c) We have $|t_1^{-e_{h1}} \cdots t_r^{-e_{hr}} x^{N-h}| < 1$ everywhere on the localizing subspace.

Recall that f is defined on some smooth formal lift P of X , for which we identify $]z[_P$ with $]z^0[_{P^0} \times A_{K,x}[0, 1]$. Lift $\bar{u}_0, \dots, \bar{u}_{N-1}$ and $\bar{g} = x^{-N}(\bar{f} - \sum_{i=0}^{N-1} \bar{f}_i x^i)$ to $u_0, \dots, u_{N-1}, g \in \Gamma(P, \mathcal{O})$. For $i = 0, \dots, N - 1$, put $f_i = u_i t_1^{e_{i1}} \cdots t_r^{e_{ir}}$. For some $c < 1$, we have $|f - gx^N - \sum_{i=0}^{N-1} f_i x^i| \leq c$ everywhere on $]z[_P$. Thanks to (b) and (c), we have $|gx^N + \sum_{i=0}^{N-1} f_i x^i| = |t_1^{e_{h1}} \cdots t_r^{e_{hr}} x^h|$ everywhere on the localizing subspace. By taking $\epsilon > \max\{c, \delta\}^{1/L}$ for $L =$

$e_{h1}b_1 + \dots + e_{hr}b_r + h$, we may ensure that $\delta \leq |t_1^{e_{h1}} \cdots t_r^{e_{hr}} x^h| = |f| < 1$ everywhere on the localizing subspace.

We next consider a condition of the form $|f_1| < |f_2|$ for f_1, f_2 (again defined on some smooth formal lift of X) having reductions $\bar{f}_1, \bar{f}_2 \in \Gamma(X, \mathcal{O})$ with $\bar{f}_2 \neq 0$ and $v_1(\bar{f}_1/\bar{f}_2) > 0$. By approximating f_1, f_2 as in the previous paragraph, after performing a suitable simple base change on Hypothesis 5.1.3 and adjusting a_i and b_i , we can find $e_{i1}, \dots, e_{ir}, j_i \in \mathbb{Z}$ for $i \in \{1, 2\}$ such that for some $\delta \in (0, 1)$, $\delta < |t_1^{e_{i1}} \cdots t_r^{e_{ir}} x^{j_i}| = |f_i| \leq 1$ everywhere on the localizing subspace. (We can only ensure strict inequality on the right in case $v_1(\bar{f}_i) > 0$.) Since $v_1(\bar{f}_1/\bar{f}_2) > 0$, we have $v_1(t_1^{e_{11}-e_{21}} \cdots t_r^{e_{1r}-e_{2r}} x^{j_1-j_2}) > 0$, and so $|f_1/f_2| = |t_1^{e_{11}-e_{21}} \cdots t_r^{e_{1r}-e_{2r}} x^{j_1-j_2}| < 1$ everywhere on the localizing subspace.

We have now shown that we can add conditions as in the definition of a localizing subspace, while maintaining the desired shape of our subspace. This proves the claim. \square

Note that Lemma 5.4.4 can equally well be stated using closed intervals $[a_j, b_j]$ instead of open intervals (a_j, b_j) , provided that we modify the definition of a localizing subspace as in Remark 2.5.8. The proof of that modified statement includes the proof of a purely characteristic p assertion. For later convenience, we extract this assertion explicitly.

Construction 5.4.5. Let $f_i : X_i^0 \rightarrow X^0$ be an exposing sequence for v^0 . We then obtain an exposing sequence for v as follows. For each index i , pick $t_1, \dots, t_r \in k(X_i^0)$ which, in some neighborhood U of the center of an exposed extension of v^0 to $k(X_i^0)$, are regular and cut out the components of $f_i^{-1}(Z^0)$ passing through the center. For $a_j, b_j \in \mathbb{Z}_{(p)}$ for $j = 1, \dots, r$ such that $a_j \leq v^0(t_j)/s_1 \leq b_j$, form the minimal normal local modification of $U \times_{X^0} X$ on which the functions $x^{b_j m}/t^m$ and $t^m/x^{a_j m}$ become regular for all $m \in \mathbb{Z}$ with $a_j m, b_j m \in \mathbb{Z}$, and let \tilde{X}_i^0 be a toroidal resolution of singularities of this (as in [24]). Then take the exposing sequence to be the maps $\tilde{X}_i^0 \rightarrow X$ indexed by integers i and tuples $(a_1, b_1, \dots, a_r, b_r) \in \mathbb{Z}_{(p)}^{2r}$.

Definition 5.4.6. Recall that we obtained M from \mathcal{E} by base change along a continuous homomorphism $\Gamma(V \times A_{K,x}[0, 1], \mathcal{O}) \rightarrow R_{v^0, [\epsilon, 1]}$ for some $\epsilon \in (0, 1)$, in which V is the intersection of $]z^0[_{P^0}$ with a strict neighborhood of $]X^0 \setminus Z^0[_{P^0}$. For a suitable choice of the localizing subspace A in Lemma 5.4.4, we have $A \subseteq V \times A_{K,x}[0, 1]$; for some closed interval J containing s_1 in its interior and some $\epsilon \in (0, 1)$, we obtain a commuting square

$$\begin{array}{ccc} \Gamma(V \times A_{K,x}[0, 1], \mathcal{O}) & \longrightarrow & R_{v^0, [\epsilon, 1]} \\ \downarrow & & \downarrow \\ \Gamma(A, \mathcal{O}) & \longrightarrow & R_{v^0, [\epsilon, 1], J} \end{array}$$

We thus obtain a base change from the Tannakian category $[\mathcal{E}_A]$ generated by \mathcal{E}_A to the Tannakian subcategory $[M_J]$ of $C_{v^0, *, J}$ generated by the restriction M_J of M from $C_{v^0, *}$ to $C_{v^0, *, J}$. Note that this functor is faithful because the map $\Gamma(A, \mathcal{O}) \rightarrow R_{v^0, [\epsilon, 1], J}$ is injective. Similarly, we obtain a base change of semisimplified categories from $[\mathcal{E}_A^{\text{ss}}]$ to $[M_J^{\text{ss}}]$. (Note that the notation M_J^{ss} is unambiguous: the image of $\mathcal{E}_A^{\text{ss}}$ is already semisimple since it generates an algebraic Tannakian category by Lemma 2.5.15. This image thus coincides with the semisimplification of M_J .)

Note that every object N of $[M_J^{\text{ss}}]$ is a subobject of an object generated from M_J^{ss} by some sequence of direct sums, tensor products, and duals. The latter can be obtained by base change from an object of $[\mathcal{E}_A^{\text{ss}}]$, by performing the same sequence of operations on $\mathcal{E}_A^{\text{ss}}$. Hence the base change functor satisfies the hypothesis of Lemma 2.5.4(c), so the automorphism group of $[M_J^{\text{ss}}]$ (for any fibre functor) may be viewed as a subgroup of the automorphism group of $[\mathcal{E}_A^{\text{ss}}]$. By similar reasoning, shrinking J does not increase the automorphism group of $[M_J^{\text{ss}}]$; we may thus assume hereafter that J is chosen so that the automorphism group $[M_J^{\text{ss}}]$ does not decrease upon shrinking J further.

With this assumption, we may assert that the automorphism group of $[M_J^{\text{ss}}]$ is insensitive to a simple base change on Hypothesis 5.1.3. By this observation plus Lemma 2.5.15, we may identify the automorphism group of $[M_J^{\text{ss}}]$ with a subgroup of the image of $\pi_1(\text{Spec } k(X)_{v_1} \times_{\text{Spec } k(X^0)} \text{Spec } k(X^0)^{\text{alg}}, \bar{x})$ under the semisimplified local monodromy representation τ of \mathcal{E} at v_1 . The ring $k(X)_{v_1} \otimes_{k(X^0)} k(X^0)^{\text{alg}}$ is a union of complete fields, and is thus henselian; we thus do not change the fundamental group by restricting to the component whose spectrum contains \bar{x} and then completing, to obtain $R_{[s_1, s_1]}$. Moreover, by Corollary 2.2.17, for J sufficiently small, the images of $\pi_1(\text{Spec } R_J, \bar{x})$ and $\pi_1(\text{Spec } R_{[s_1, s_1]}, \bar{x})$ under τ coincide.

Let $\tilde{\tau}$ denote the restriction of τ to $\pi_1(\text{Spec } R_J, \bar{x})$. We now have an identification of the automorphism group of $[M_J^{\text{ss}}]$ with a subgroup of the image of $\tilde{\tau}$. We will show momentarily that the p -parts of these two groups coincide for J sufficiently small; see Lemma 5.4.8.

We need the following compatibility between the previous construction and the construction relating Artin-Schreier characters to Dwork modules.

Lemma 5.4.7. *With notation as in Definition 5.4.6, suppose ψ is a subrepresentation of $\tilde{\tau}$ isomorphic to the Artin-Schreier character defined by a prepared parameter $u \in R_J$. Then the corresponding subquotient of M_J in $C_{v^0, *, J}$ is isomorphic to M_u .*

Proof. Suppose first that ψ is the trivial representation, in which case we must check that the subquotient of M_J in $C_{v^0, *, J}$ under consideration is trivial. This holds because the automorphism group of $[M_J^{\text{ss}}]$ is a subgroup of the image of $\tilde{\tau}$, so any trivial subrepresentation of $\tilde{\tau}$ gives rise to a trivial object in $[M_J^{\text{ss}}]$.

Let Z' be the union of Z with the zero locus of x . Suppose next that we can produce, for each prepared Artin-Schreier parameter $u \in R_J$, an F -isocrystal \mathcal{E}_u of rank 1 on $X \setminus Z'$ overconvergent along Z' , for which $\tilde{\tau}$ is the Artin-Schreier character defined by u , and $M_J \cong M_u$. Then we may apply the previous paragraph with \mathcal{E} replaced by $\mathcal{E}_u^\vee \otimes \mathcal{E}$ to conclude. (Note that although \mathcal{E}_u is not defined on the zero locus of x , the construction of M_J from Definition 5.4.6 still makes sense for \mathcal{E}_u .)

It thus remains to produce \mathcal{E}_u . Since $u \in k(X^0)[x, x^{-1}]$, we may perform a simple base change on Hypothesis 5.1.3 to ensure that $u \in \Gamma(X^0 \setminus Z^0, \mathcal{O})[x, x^{-1}]$. We may then form a finite normal cover X' of X with function field $k(X)[z]/(z^p - z - u)$. The pushforward of the trivial isocrystal on X' gives an overconvergent F -isocrystal \mathcal{F} on $X \setminus Z'$ carrying an action of $\mathbb{Z}/p\mathbb{Z} \cong \text{Gal}(k(X')/k(X))$. Since \mathcal{F} becomes globally constant on X' , the local monodromy representation of \mathcal{F} produced by Definition 2.5.14 coincides with the restriction of the global monodromy representation produced by Crew [7, Theorem 2.1] (see also [22,

Theorem 2.3.4]). The latter is the regular representation of $\text{Gal}(k(X')/k(X))$, which splits into one-dimensional characters; correspondingly, \mathcal{F} splits into rank 1 isocrystals, one of which has monodromy representation equal to the Artin-Schreier character defined by the parameter u . We call this module \mathcal{E}_u .

It remains to verify that for this choice of \mathcal{E}_u , we have $M_J = M_u$. This follows from the description of M_u given in Lemma 5.3.5(b), together with the observations that

$$\pi^{-p}((1 + \pi z)^p - (1 - p\pi\tilde{u})) \equiv z^p - z - u \pmod{\mathfrak{m}_K}$$

(this matches up the Artin-Schreier extensions) and

$$\pi^{-1}(\zeta_p(1 + \pi z) - (1 + \pi z)) \equiv 1 \pmod{\mathfrak{m}_K}$$

(this matches up the $\mathbb{Z}/p\mathbb{Z}$ -torsor structures). \square

Lemma 5.4.8. *With notation as in Definition 5.4.6, for J sufficiently small, the automorphism group of $[M_J^{\text{ss}}]$ is a subgroup of $\tilde{\tau}$ having the same p -Sylow subgroup.*

It is possible to show that the groups actually match, but we will not need that stronger result.

Proof. As in Definition 5.4.6, we take J small enough so that neither the automorphism group of $[M_J^{\text{ss}}]$ nor the image of $\pi_1(\text{Spec } R_J, \bar{x})$ under $\tilde{\tau}$ decreases upon shrinking J . Let G be the image of $\tilde{\tau}$, and let H be the automorphism group of $[M_J^{\text{ss}}]$, identified with a subgroup of G . By Definition 2.5.6, G has a unique p -Sylow subgroup P , and G/P is abelian of order prime to p .

Note that it is harmless to make a tame alteration on Hypothesis 5.1.3 before checking the claim: this tame alteration can reduce G and H but not P , so the difference between P and $H \cap P$ cannot decrease. We may thus reduce to the case where $G = P$ is a p -group.

Suppose now that H is a proper subgroup of G . Then H is contained in a maximal proper subgroup of G ; since G is a p -group, any maximal proper subgroup is normal of index p . Thus there is an order p abelian character of G whose restriction to H is trivial; however, by Lemma 5.4.7, this would produce a prepared parameter u which corresponds to a nontrivial Artin-Schreier character of R_J , but for which M_u is trivial. As noted in Definition 5.3.2, we must have $v_s(u) \leq 0$ for some $s \in J$, and likewise after shrinking J ; we must thus have $v_{s_1}(u) \leq 0$. Since s_1 is not in the divisible closure of Γ_{v^0} , we must in fact have $v_{s_1}(u) < 0$; however, this contradicts Lemma 5.3.5(d). We thus have $H = G$, as desired. \square

5.5 Terminal decompositions

Definition 5.5.1. We say that M is *terminally presented* if the following conditions hold.

- (a) For $i = 1, \dots, \text{rank}(M)$, $b_i(M, \alpha, s)$ is affine-linear on $[0, s_0]$ with slope at most 1.
- (b) For $i = 1, \dots, \text{rank}(M) - 1$, either $b_i(M, \alpha, s) = b_{i+1}(M, \alpha, s)$ identically for $s \in [0, s_0]$, or $b_i(M, \alpha, s) > b_{i+1}(M, \alpha, s)$ identically for $s \in [0, s_0]$.

This property is stable under performing a base change on Hypothesis 5.1.3, because the function $b_i(M, \alpha, s) - s$ is preserved up to a left shift of the domain.

Lemma 5.5.2. *After performing a suitable base change on Hypothesis 5.1.3 (depending on M), we can force M to become terminally presented.*

Proof. This holds by Proposition 4.6.4 and Corollary 4.7.7. \square

One might hope that having $\text{End}(M)$ being terminally presented would allow us to decompose M according to $b_i(M, \alpha, s)$ in the manner of Theorem 3.1.6. It is the source of some complication in the argument that only a weaker decomposition result can be established.

Lemma 5.5.3. *With notation as in Definition 5.4.6, suppose that $\alpha_{0,e^{-s}} \geq \alpha$ for $s \in J$, and that M is terminally presented.*

- (a) *Each trivial subobject of M_J induces a (unique) trivial subobject of M in $C_{v^0,*,**,*,\alpha}$.*
- (b) *For J sufficiently small, there is a direct summand of M_J whose Jordan-Hölder constituents are exactly the trivial Jordan-Hölder constituents of M_J .*

Proof. Pick some $s_2 \in (0, s_0)$ such that $(0, s_2) \cap J \neq \emptyset$, and make a scale-preserving base change on Hypothesis 5.1.3 to ensure that $\alpha_{0,e^{-s_2}} \geq \alpha$. By Theorem 3.1.6, for each ρ sufficiently close to 1, we may decompose M_ρ over $A_{F_\rho}(\rho^{s_2}, 1)$ as $M_{\rho,0} \oplus M_{\rho,1}$, such that for $r = -\log \rho$ and $s \in J$, we have $f_i(M_{\rho,0}, rs) = rs$ and $f_i(M_{\rho,1}, rs) > rs$ for all i . From Definition 2.5.14, there exists a finite étale extension B of A over which \mathcal{E}_A becomes unipotent; tensoring with $R_{v^0, [\epsilon, 1], J}$, we obtain a finite étale cover of $A_{F_\rho}(\rho^J)$ on which $M_{\rho,0}$ becomes unipotent. By Proposition 3.2.5, $M_{\rho,0}$ becomes unipotent after pullback along the cover $h : A_{F_\rho}(\rho^{s_2/m}, 1) \rightarrow A_{F_\rho}(\rho^{s_2}, 1)$ defined by $x \mapsto x^m$ for some positive integer m coprime to p .

This implies (e.g., by [13, Lemma 4.29]) that if we formally adjoin $\log(x)$ to the structure sheaf of $A_{F_\rho}(\rho^{s_2/m}, 1)$, then $h^*M_{\rho,0}$ acquires a full basis of horizontal sections. Let V be the F_ρ -vector space spanned by these horizontal sections; then V carries an action of the Galois group G of the cover h . Splitting V into isotypical spaces for the G -action induces a decomposition of $M_{\rho,0}$; the isotypical space for the trivial representation induces a decomposition $M_{\rho,0} \cong M'_{\rho,0} \oplus M''_{\rho,0}$ in which the Jordan-Hölder constituents of $M'_{\rho,0}$ consist precisely of the trivial Jordan-Hölder constituents of $M_{\rho,0}$. (Compare [13, Proposition 4.37].)

Let \mathbf{v} be a horizontal section of M in $C_{v^0,*,J}$. The image of \mathbf{v} in M_ρ over $A_{F_\rho}(\rho^J)$ must belong to $M'_{\rho,0}$; it must then extend all the way over $A_{F_\rho}(\rho^{s_2}, 1)$. By Lemma 4.9.5, \mathbf{v} lifts to a horizontal section of M in $C_{v^0,*,(0,s_2)\cup J}$. We may deduce (a) from this by glueing over all choices of s_2 .

We proceed now to (b). By shrinking J , we may assume that its left endpoint is in $v^0(\ell)$; let J' be the translate of J with left endpoint 0. By performing a center-preserving base change on Hypothesis 5.1.3 that shifts J to J' , we obtain an equivalence of categories between $C_{v^0,*,J}$ and $C_{v^0,*,J'}$. (More precisely, if we choose $\bar{h} \in \ell$ with $v^0(\ell)$ being the left endpoint of J , then the substitution $x \mapsto x[\bar{h}]$ induces an isomorphism $R_{v^0,I,J} \rightarrow R_{v^0,I,J'}$ for

any interval $I \subseteq (0, 1)$.) Let $\tilde{M} \in C_{v^0, *, J'}$ be the image of M_J under this equivalence, so that we obtain a decomposition $\tilde{M}_\rho \cong \tilde{M}_{\rho,0} \oplus \tilde{M}_{\rho,1}$. We may then combine this decomposition with the decomposition of \tilde{M} given by Lemma 4.3.4 (by invoking [23, Lemma 2.3.1] and Remark 4.9.11); the resulting decomposition of \tilde{M} reflects back to a decomposition of M_J lifting the decomposition $M_\rho \cong M_{\rho,0} \oplus M_{\rho,1}$ for each ρ sufficiently close to 1.

Let N be the direct summand of M_J lifting $M_{\rho,0}$. Suppose P is a Jordan-Hölder constituent of M_J which becomes trivial after pullback along $x \mapsto x^m$ for some positive integer m coprime to p . By Lemma 3.1.8, $b_1(P, s) = s$ for all $s \in J$; consequently, P must be a constituent of N .

On the other hand, we claim that after pullback along $x \mapsto x^m$ for some positive integer m coprime to p , N becomes unipotent. In other words, after a suitable tame alteration on Hypothesis 5.1.3, N becomes unipotent. By invoking Remark 5.4.3, we can choose the tame alteration so that $\tau(I_1) = \tau(W'_1)$ is a p -group. By Lemma 5.4.8, by making J sufficiently small, we can ensure that the images of τ and $\tilde{\tau}$ coincide, so the latter is also a p -group. Suppose by way of contradiction that N has still not become unipotent. By Lemma 2.5.1, after possibly replacing K by a finite extension, we would find a nontrivial one-dimensional subquotient P of (the pullback along the alteration of) $N \oplus \text{End}(N)$ such that for some nonnegative integer h , $P^{\otimes p^h}$ corresponds to a nontrivial Artin-Schreier character of I_1 via Lemma 5.4.7. But $b_1(N, \alpha, s) = s$ for all $s \in J$, so by Corollary 5.3.6, after shrinking J this character must become trivial, a contradiction. (Compare the proof of Lemma 5.6.2.)

By splitting N into isotypical components for the Galois group of the cover $x \mapsto x^m$ (as above), we obtain a direct summand of M_J whose Jordan-Hölder constituents are exactly the trivial Jordan-Hölder constituents of M_J . This proves (b). \square

Corollary 5.5.4. *With notation as in Definition 5.4.6, suppose that $\alpha_{0,e^{-s}} \geq \alpha$ for $s \in J$, and that $\text{End}(M)$ is terminally presented. Then for J sufficiently small, we have the following.*

- (a) *Any irreducible subquotient of M in $C_{v^0, *, J}$ is isomorphic to an irreducible subobject.*
- (b) *Each irreducible subobject of M in $C_{v^0, *, J}$ induces a (unique) subobject of M in $C_{v^0, **, \alpha}$.*

Proof. By Lemma 5.5.3(b), for J sufficiently small, $\text{End}(M_J)$ splits as a direct sum $N_0 \oplus N_1$ in which the Jordan-Hölder constituents of N_0 are exactly the trivial Jordan-Hölder constituents of $\text{End}(M_J)$. Let P be any irreducible subquotient of M_J . Then $P^\vee \otimes M_J$ occurs as a subquotient of $\text{End}(M_J)$, so it also splits as a direct sum $Q_0 \oplus Q_1$ in which the Jordan-Hölder constituents of Q_0 are exactly the trivial Jordan-Hölder constituents of $P^\vee \otimes M_J$. The image of $P \otimes Q_0$ under the trace map $P \otimes P^\vee \otimes M_J \rightarrow M_J$ is a nonzero subobject of M_J whose Jordan-Hölder constituents are all isomorphic to P ; this proves (a).

By a similar argument, P also occurs as a nonzero quotient of M_J . Composing a surjection $M_J \rightarrow P$ with an injection $P \rightarrow M_J$ gives an endomorphism of M_J with image isomorphic to P . This suffices to prove (b) because Lemma 5.5.3 applied to $\text{End}(M)$ transfers this endomorphism to $C_{v^0, **, \alpha}$. \square

5.6 Terminal unipotence

We are now equipped to attack Theorem 5.0.1 by reducing to consideration of Artin-Schreier characters.

Definition 5.6.1. Let M_{**} be the restriction of M to $C_{v^0,*,**, \alpha}$. We say M is *terminally unipotent* if M_{**} is a successive extension of trivial objects in $C_{v^0,*,**, \alpha}$.

Lemma 5.6.2. *After a suitable sequence of base changes and tame alterations on Hypothesis 5.1.3, and after replacing K by a finite extension, one of the following conditions becomes true.*

- (a) M is terminally unipotent.
- (b) We can find a nontrivial subobject N of $M_{**} \oplus \text{End}(M_{**})$ of rank 1, such that $N^{\otimes p^h}$ is trivial for some positive integer h .

Proof. By Lemma 5.5.2, after a base change on Hypothesis 5.1.3, we may assume that $\text{End}(M)$ is terminally presented. Note that this condition is preserved by further base changes and tame alterations on Hypothesis 5.1.3.

Pick $s_1 > 0$ not in the divisible closure of the value group Γ_{v^0} of v^0 . After a scale-preserving base change on Hypothesis 5.1.3, we may assume that $\alpha_{0,e-s_2} \geq \alpha$ for some $s_2 \in (s_1, s_0)$. We may then set notation as in Definition 5.4.6, shrinking J if necessary to force $J \subseteq [0, s_2]$. Note that all of these conditions are preserved by further *simple* base changes and tame alterations.

By Remark 5.4.3, after a simple base change and tame alteration on Hypothesis 5.1.3, we have $\tau(I_1) = \tau(W'_1)$, which is a p -group. By shrinking J again and invoking Lemma 5.4.8, we may ensure that $\tilde{\tau}(\pi_1(\text{Spec } R_J, \bar{x})) = \tau(I_1)$. By Lemma 2.5.1, after possibly replacing K by a finite extension, either $\tilde{\tau}$ is trivial, or $\tilde{\tau} \oplus \text{End}(\tilde{\tau})$ has a nontrivial one-dimensional subrepresentation ψ such that $\psi^{\otimes p^h}$ is trivial for some positive integer h . In the former case, Corollary 5.5.4 ensures that M_{**} is unipotent. In the latter case, Corollary 5.5.4 ensures firstly that ψ corresponds to a subobject (not just a subquotient) N of M_J , secondly that N lifts to a subobject of M_{**} , and thirdly that $N^{\otimes p^h}$ is trivial as a subobject of $M_{**}^{\otimes p^h}$. \square

We now use what we have learned from the machinery of differential modules to show that we can always reduce consideration of an Artin-Schreier extension to the very special shape permitted by an Artin-Schreier alteration on Hypothesis 5.1.3. Without this analysis, we would be unable to simplify \mathcal{E} by passing up Artin-Schreier extensions while maintaining smoothness of the map $X \rightarrow X^0$ at the center of v .

Lemma 5.6.3. *Suppose that N is a rank 1 object of the Tannakian subcategory of $C_{v^0,*,**, \alpha}$ generated by M_{**}^{ss} , such that $N^{\otimes p^h}$ is trivial for some nonnegative integer h . Then after a suitable sequence of base changes and Artin-Schreier alterations on Hypothesis 5.1.3, N becomes trivial.*

Proof. It suffices to treat the case $h = 1$, as the general case follows by induction on h by applying the base case to $N^{\otimes p^{h-1}}$. Reduce to the case where M is terminally presented, then set notation as in the proof of Lemma 5.6.2. Then N corresponds to an Artin-Schreier representation ψ of $\pi_1(\text{Spec } R_J, \bar{x})$ for some J , which after a simple base change on Hypothesis 5.1.3 we may assume is generated by a prepared parameter $u \in R_J$. By Lemma 5.4.7, N is isomorphic as an element of $C_{v^0,*,J}$ to the Dwork module M_u .

By Proposition 5.3.7, we may choose $u \in \ell\langle x \rangle$ prepared so that $N \cong M_u$ in $C_{v^0,*,**,*,\alpha}$. (This requires combining parts (b) and (c) of Proposition 5.3.7, as in the proof of Proposition 5.3.7(d).) Moreover, the terminal slope of $b_1(M_u, \alpha, s)$ is equal to 0 unless $b_1(M_u, \alpha, s) = s$ identically (only *a priori* in a neighborhood of s_0 , but we assumed that M is terminally presented). In the latter case, by Corollary 5.3.6, ψ is trivial. In the former case, write $u = \sum_{i=1}^h u_i x^i$; by Lemma 5.3.5(d), we have $v^0(u_1) + s \leq v^0(u_i) + is$ for $i > 1$ and $s \in J$. We also have $v^0(u_1) + s \leq s$ for $s \in J$, so $v^0(u_1) \leq 0$. By a center-preserving base change on Hypothesis 5.1.3, we may force the previous inequality to hold for all $s \geq 0$, while preserving the property that $\alpha_{0,e^{-s}} \geq \alpha$ for some $s > 0$.

Put $u' = u_1 x$; then $b_1(M_u, s) \geq b_1(M_u \otimes M_{u'}^\vee, s)$ for $s \geq 0$. In this inequality, the left side is constant and greater than or equal to s ; by contrast, the right side has negative slope until it equals s and then is identically equal to s thereafter. It follows that for $s > 0$, we have $b_1(M_u, s) > b_1(M_u \otimes M_{u'}^\vee, s)$ unless both sides are equal to s .

By a center-preserving base change on Hypothesis 5.1.3, we can force $b_1(M_u, s) > b_1(M_u \otimes M_{u'}^\vee, s)$ for $s \geq 0$, while preserving the property that $\alpha_{0,e^{-s}} \geq \alpha$ for some $s > 0$. Taking $s = 0$, we find $v^0(u_1) < v^0(u_i)$, or $v^0(u_i/u_1) > 0$. By a simple base change on Hypothesis 5.1.3, we can force u_1 to be a p -th power in $k(X^0)$, and we may force u_i/u_1 to belong to the maximal ideal of \mathcal{O}_{X^0, z^0} for $i > 1$. Since $v^0(u_1) \leq 0$, we may also force $u_1^{-1/p}$ to belong to $\Gamma(X^0, \mathcal{O})$. Perform an Artin-Schreier alteration on Hypothesis 5.1.3 with $g = u_1^{-1/p}$ and $a_i = u_i/u_1$ for $i > 1$.

At this point, the representation ψ is now trivial. By Lemma 5.4.7 again, N is trivial as an object in $C_{v^0,*,J} = C_{v^0,*,J,\alpha}$. Perform a center-preserving base change on Hypothesis 5.1.3 to force 0 into J ; then by Proposition 5.3.7(c), N is also trivial in $C_{v^0,*,**,*,\alpha}$. \square

Putting together the two previous arguments yields the following.

Proposition 5.6.4. *After a sequence of base changes, tame alterations, and Artin-Schreier alterations on Hypothesis 5.1.3, and after possibly replacing K by a finite extension, M becomes terminally unipotent.*

Proof. We proceed by descending induction on the number of trivial Jordan-Hölder constituents of $M_{**} \oplus \text{End}(M_{**})$. The maximum value of this number occurs when $M_{**} \oplus \text{End}(M_{**})$ is itself terminally unipotent, in which case M is as well.

By Lemma 5.6.2 (applied after possibly replacing K by a finite extension), we can perform a sequence of base changes and tame alterations on Hypothesis 5.1.3, after which either M is terminally unipotent, or there exists a nontrivial subobject N of $M_{**} \oplus \text{End}(M_{**})$ of rank 1, such that $N^{\otimes p^h}$ is trivial for some positive integer h . In the former case, we are done.

In the latter case, by Lemma 5.6.3, we can perform a sequence of base changes and Artin-Schreier alterations on Hypothesis 5.1.3 to make N trivial. Then we may apply the induction hypothesis to conclude. \square

5.7 Completion of the proof

To finish the proof of Theorem 5.0.1, we now show that terminal unipotence of M implies local semistable reduction at v . Beware that we cannot apply Proposition 2.5.16 directly, since that result assumes local semistable reduction at v .

Proposition 5.7.1. *Suppose that M is terminally unipotent. Then \mathcal{E} admits local semistable reduction at v .*

Proof. Since M is terminally unipotent, we may make a base change on Hypothesis 5.1.3 to reduce to the case where the restriction of M to $C_{v^0,*,*,\alpha}$ (rather than $C_{v^0,*,**,*,\alpha}$) is a successive extension of trivial objects. Note that this condition is preserved by further base changes on Hypothesis 5.1.3.

We then set notation as in the proof of Lemma 5.6.2. Namely, pick $s_1 > 0$ not in the divisible closure of the value group Γ_{v^0} of v^0 . After a scale-preserving base change on Hypothesis 5.1.3, we may assume that $\alpha_{0,e-s_2} \geq \alpha$ for some $s_2 \in (s_1, s_0)$. We may then set notation as in Definition 5.4.6, shrinking J if necessary to force $J \subseteq [0, s_2]$. By a suitable sequence of base changes and tame alterations on Hypothesis 5.1.3, we can force $\tau(I_1) = \tau(W'_1)$. The latter group must be trivial, or else as in the proof of Lemma 5.6.2, we could find a nontrivial subobject N of M_{**} , contradicting our assumption that M is terminally unipotent.

In other words, we are now in the case where the semisimplified local monodromy representation of \mathcal{E} at v_1 is trivial. We now apply Proposition 2.5.16 to v_1 , using an exposing sequence generated by Construction 5.4.5. This implies that after a simple base change on Hypothesis 5.1.3, we can choose:

- $t_1, \dots, t_r \in \Gamma(X^0, \mathcal{O})$ which cut out the components of Z^0 passing through z^0 ;
- $a_i, b_i \in \mathbb{Z}_{(p)}$ for $i = 1, \dots, r$ such that $v^0(t_i)/b_i < s_1 < v^0(t_i)/a_i$;
- a positive integer m with $a_i m, b_i m \in \mathbb{Z}$ for $i = 1, \dots, r$;
- a desingularization \tilde{Y} of the closure in $X \times_k \mathbb{A}_k^{2r}$ of the graph of the rational map $X \dashrightarrow \mathbb{A}_k^{2r}$ defined by $x^{b_i m}/t_i^m$ and $t_i^m/x^{a_i m}$ for $i = 1, \dots, r$ (which exists by toroidal resolution of singularities [24]);

such that \mathcal{E} is log-extendable to (\tilde{Y}, \tilde{W}) , for \tilde{W} the inverse image of Z in \tilde{Y} . By one direction of [17, Theorem 6.4.5], \mathcal{E} has unipotent local monodromy along every divisorial valuation of $k(X)$ centered on \tilde{Y} .

To extract the desired conclusion, we argue as in [19]. Fix some local coordinates of X at z including x, t_1, \dots, t_r , and use them to define the derivations $\partial_0 = \frac{\partial}{\partial x}$ and $\partial_i = \frac{\partial}{\partial t_i}$ for

$i = 1, \dots, r$. For $R = (\rho_0, \dots, \rho_r) \in (0, 1)^{r+1}$, define $T(\mathcal{E}, R)$ using the derivations $\partial_0, \dots, \partial_r$, as in [19, Definition 4.2.1]. Put

$$D = \{(u_0, \dots, u_r) \in \mathbb{R}^{r+1} : u_0, \dots, u_r > 0, \quad u_1 + \dots + u_r = 1\}.$$

Define the function $h : D \cap \mathbb{Q}^{r+1} \rightarrow [0, +\infty) \cap \mathbb{Q}$ by requiring that for $U = (u_0, \dots, u_r) \in D \cap \mathbb{Q}^{r+1}$, for $\rho \in (0, 1)$ sufficiently close to 1, we have $T(\mathcal{E}, (\rho^{u_0}, \dots, \rho^{u_r})) = \rho^{h(U)}$. (The quantity $h(U)$ exists by [19, Theorem 4.4.7].) The function h then has the following properties.

- (Monotonicity) The function $h(U)$ is nonincreasing in u_0 (as in [20, §11.4]).
- (Relation to unipotent monodromy) Let v_U be the toric valuation with $v_U(x) = u_0$ and $v_U(t_i) = u_i$ for $i = 1, \dots, r$. Then $h(U) = 0$ if and only if \mathcal{E} acquires unipotent local monodromy along v_U after pulling back along some tamely ramified cover branched over the zero loci of x, t_1, \dots, t_r [19, Proposition 5.3.4].

By the relation to unipotent monodromy, we have $h(U) = 0$ for all $U \in D \cap \mathbb{Q}^{r+1}$ such that $b_i^{-1}u_i \leq u_0 \leq a_i^{-1}u_i$ for $i = 1, \dots, r$. By monotonicity, we also have $h(U) = 0$ for all $U \in D \cap \mathbb{Q}^{r+1}$ such that $b_i^{-1}u_i \leq u_0$ for $i = 1, \dots, r$. In particular, since $s_1 \leq v(x)$, this includes a neighborhood of the point $(v(x), v(t_1), \dots, v(t_r))$.

By a suitable toroidal blowup in x, t_1, \dots, t_r , we obtain a local alteration $f_1 : X_1 \rightarrow X$ around v , such that (X_1, Z_1) is a smooth pair for $Z_1 = f_1^{-1}(Z)$, and each of the components of Z_1 corresponds to a divisorial valuation of the form v_U for some $U \in D$ with $h(U) = 0$. For m a positive integer prime to p , let $X_{2,m}$ be the finite cover of X_1 which is tamely ramified of degree m along each component of Z_1 ; then $f_{2,m} : X_{2,m} \rightarrow X$ is again a local alteration around v such that $(X_{2,m}, Z_{2,m})$ is a smooth pair for $Z_{2,m} = f_{2,m}^{-1}(Z)$. By the relation to unipotent monodromy, for m sufficiently divisible, $f_{2,m}^*\mathcal{E}$ admits unipotent local monodromy along each component of $Z_{2,m}$. By the other direction of [17, Theorem 6.4.5], $f_{2,m}^*\mathcal{E}$ is log-extendable to $(X_{2,m}, Z_{2,m})$ as desired. \square

To conclude, we summarize how all of the ingredients in this section come together to give a proof of Theorem 5.0.1.

Proof of Theorem 5.0.1. It suffices to show that under Hypothesis 5.1.1, \mathcal{E} admits local semistable reduction at v . During the course of the proof, we may at any time replace X by a (not necessarily separable) local alteration of X around v , pulling \mathcal{E} back accordingly. This means on one hand that by Lemma 5.1.2, we may reduce to the case where Hypothesis 5.1.3 holds, and on the other hand, we may perform base changes, tame alterations, and Artin-Schreier alterations on Hypothesis 5.1.3. Using such operations, by Proposition 5.6.4, we may arrive at the situation where M is terminally unipotent. By Proposition 5.7.1, \mathcal{E} admits local semistable reduction at v as desired. \square

A Appendix

In this appendix, we include two technical corrections, one to [17] and one to [18]. We also discuss the prospect of removing the hypothesis that the coefficient field K is discretely valued.

A.1 A technical correction

This subsection reports and corrects an error in [17], kindly pointed out to us by Atsushi Shiho. To lighten notation, in this subsection we make all citations directly to [17] without saying so each time.

The error occurs in two places. In Proposition 3.5.3, the erroneous statements are the last two sentences of the proof: since the strict neighborhood V' may not have integral reduction, one cannot necessarily embed it isometrically into a field. A similar error occurs in Lemma 3.6.2, because X need not have integral reduction.

In the case of Lemma 3.6.2, this is most easily corrected by simply requiring X to have integral reduction, and making the same requirement in Proposition 3.6.9. This suffices because the only invocation of Lemma 3.6.2 is in Proposition 3.6.9, and all invocations of Proposition 3.6.9 occur in cases where X has integral reduction.

In the case of Proposition 3.5.3, the only invocation occurs in Lemma 5.1.1, in which we may assume $b > 0$. Under this extra hypothesis, we may correct the proof of Proposition 3.5.3 as follows. (Thanks to Shiho for feedback on this correction.)

Retain notation from the proof of Proposition 3.5.3. We have from the correct part of the proof that $H_{V_\lambda}^0(V_\lambda \times A_K^n[b, c], \mathcal{E}) \neq 0$. Take the \mathcal{O} -span of this set; it is a submodule of \mathcal{E} stable under ∇ . Restrict to $V_\lambda^0 \times A_K^n[b, c]$, for V_λ^0 the nonlogarithmic locus of V_λ ; by Proposition 3.3.8, this restriction extends to a log- ∇ -submodule \mathcal{F} of \mathcal{E} . On $(V_\lambda^0 \cap]X[) \times A_K^n[b, c]$, \mathcal{F} is constant relative to $V_\lambda^0 \cap]X[$; we may infer the same conclusion on $]X[\times A_K^n[b', c']$ for any closed subinterval $[b', c']$ of (b, c) , via Corollary 3.4.5.

In other words, the whole proof reduces to the case when \mathcal{E} is constant, not just unipotent, on $]X[\times A_K^n[b, c]$. In this case, we must show that \mathcal{E} is also constant on $V_\lambda \times A_K^n[b, c]$. We first prove that $g_\lambda : \pi_1^* H_{V_\lambda}^0(V_\lambda \times A_K^n[b, c], \mathcal{E}) \rightarrow \mathcal{E}$ is surjective for some λ . For $\mathbf{v} \in \Gamma(V \times A_K^n[d, e], \mathcal{E})$, the correct part of the proof of Proposition 3.5.3 shows that the sequence $D_l(\mathbf{v})$ converges on $V_\lambda \times A_K^n[b, c]$ for some λ . Moreover, as in Proposition 3.4.3, on $]X[\times A_K^n[b, c]$ we have the identity

$$\mathbf{v} = \sum_{J \in \mathbb{Z}^n} t_1^{j_1} \cdots t_n^{j_n} f(t_1^{-j_1} \cdots t_n^{-j_n})$$

(using the hypothesis that $b \neq 0$), so the cokernel of g_λ has no support on $]X[\times A_K^n[b, c]$. On $V_\lambda \times A_K^n[b, c]$, the support of this cokernel is a closed analytic subspace not meeting $]X[\times A_K^n[b, c]$; by the maximum modulus principle, it also fails to meet $V_{\lambda'} \times A_K^n[b, c]$ for some $\lambda' \in (1, \lambda]$. Hence for suitable λ , g_λ is surjective.

Over $V_\lambda^0 \times A_K^n[b, c]$, the map g_λ is automatically a morphism in $\text{LNM}_{V_\lambda^0 \times A_K^n[b, c]}$ because there are no logarithmic singularities; hence Proposition 3.2.20 implies that the restriction of \mathcal{E} to $V_\lambda^0 \times A_K^n[b, c]$ is constant over V_λ^0 .

To finish the proof that \mathcal{E} is constant, it suffices to do so after extending scalars from K to a finite extension. By doing so, we may ensure that there exists a K -rational point $x \in A_K^n[b, c]$. Let \mathcal{E}_0 be the restriction of \mathcal{E} to $V_\lambda \times \{x\}$, identified with V_λ . We then have an isomorphism of $\pi_1^*\mathcal{E}_0$ with \mathcal{E} over $V_\lambda^0 \times A_K^n[b, c]$ (because \mathcal{E} is constant over V_λ^0); by Proposition 3.3.8 (applied to the graph of the isomorphism inside of $\pi_1^*\mathcal{E}_0 \oplus \mathcal{E}$), this extends to an isomorphism $\pi_1^*\mathcal{E}_0 \cong \mathcal{E}$ over $V_\lambda \times A_K^n[b, c]$.

We also note that Shiho has generalized the results of §3 in [33], using somewhat different arguments. Hence one can also avoid the aforementioned errors (and gain some simplification in the proofs) simply by invoking [33] instead.

A.2 Another technical correction

The referee of the present paper, and independently Atsushi Shiho, have pointed out some problems with the handling of imperfect base fields in [18]. We remedy these here.

We first note that [18, Theorem 3.1.3] is only a correct statement of de Jong's alterations theorem in case k is perfect; otherwise, we only may conclude that \overline{X}_1 is regular, not necessarily smooth over k .

As a result, the statements of [18, Conjecture 3.2.5] and its variants are not tenable. The simplest way to correct the statement, as we have done in § 2.4 of this paper, is to weaken the definition of semistable reduction [18, Definitions 3.2.4 and 3.4.1] to require the quasiresolution and logarithmic extension only to be defined over k^{-q^n} for some nonnegative integer n (with K then replaced with $K^{\sigma_K^n}$).

With this change, the fact that semistable reduction for a given isocrystal over k is equivalent to semistable reduction over k^{perf} becomes an immediate consequence of the fact that a pair (X, Z) of k -varieties becomes smooth over k^{perf} if and only if it becomes smooth over $k^{p^{-n}}$ for some n (and hence for all sufficiently large n). As a result, [18, Proposition 3.2.6] remains correct (using the new definition), with the given proof applied only for k perfect.

We similarly modify the definition of local semistable reduction [18, Definitions 3.3.1 and 3.4.3] to allow replacing k with k^{-q^n} for some positive integer n before constructing the quasiresolution. The proof of [18, Proposition 3.3.4], and by extension [18, Proposition 3.4.5], carry over with minor modifications; alternatively, one may simply invoke the amended [18, Proposition 3.2.6] to reduce to the case of k perfect.

The one other argument that is affected is the proof of [18, Lemma 4.3.1]. By the amended [18, Proposition 3.2.6], we may assume k is algebraically closed. With this assumption, even though the quasiresolution (f_1, j_1) is now only defined over $\ell^{q^{-n}}$ for some n , the recipe for construction the quasiresolution (f_2, j_2) carries over without change.

A.3 Removing the discreteness hypothesis

Throughout this series of papers, we have been forced in various places to assume that K is not an arbitrary complete nonarchimedean field, but rather a complete discretely valued field. This may ultimately not be necessary, as the approach of André [1] and Mebkhout

[27] to the p -adic local monodromy theorem can be extended to the nondiscrete case. (It is unclear whether our approach using Frobenius slope filtrations [12] can be so extended.)

It is thus worth cataloging exactly where the discreteness hypothesis is being used in our work, and pointing out what may potentially be done to relax it.

- In [17], the only use of the discreteness hypothesis is to exploit Shiho’s theory of convergent log-isocrystals. One could make an *ad hoc* construction for the case of log-schemes associated to smooth pairs, without relying on the discreteness hypothesis.
- In [18], the discreteness hypothesis is used essentially in Theorem 4.2.1, a result on the full faithfulness of restriction from a category of overconvergent F -isocrystals to the corresponding category of convergent F -isocrystals. The restriction ultimately occurs in a local problem, whose solution depends on the p -adic local monodromy theorem (see for instance [20, Theorem 20.3.5]); it may be possible to use Christol-Mebkhout decomposition theory instead.
- In [19], discreteness is needed because the proof of local semistable reduction at monomial valuations ultimately relies on our extension of the p -adic local monodromy theorem to fake annuli [16], which uses Frobenius slope filtrations. However, it should be possible to argue using Mebkhout’s approach to the local monodromy theorem, by using Christol-Mebkhout decomposition theory to reduce to the case of a connection of rank 1. (The reader may detect echoes of that strategy in this paper, especially in § 5.6.) The role of decomposition theory in this case would be played by the higher-dimensional decomposition theory we gave with Xiao in [23].
- In this paper, the discreteness hypothesis is used starting in Definition 4.1.1, in order to use the formalism of analytic rings from [14]. It should be possible to circumvent this hypothesis with an improved formalism.

References

- [1] Y. André, Filtrations de type Hasse-Arf et monodromie p -adique, *Invent. Math.* **148** (2002), 285–317.
- [2] V.G. Berkovich, *Spectral theory and analytic geometry over non-archimedean fields* (translated by N.I. Koblitz), Math. Surveys and Monographs 33, Amer. Math. Soc., Providence, 1990.
- [3] P. Berthelot, Cohomologie rigide et cohomologie rigide à support propre. Première partie, Prépublication IRMAR 96-03, available at <http://www.math.univ-rennes1.fr/~berthelo>.
- [4] S. Boucksom, C. Favre, and M. Jonsson, Valuations and plurisubharmonic singularities, *Publ. RIMS, Kyoto Univ.* **44** (2008), 449–494.

- [5] D. Caro, Stabilité de l’holonomie sur les variétés quasi-projectives, arXiv preprint 0810.0304v3 (2009).
- [6] D. Caro and N. Tsuzuki, Overholonomicity of overconvergent F -isocrystals over smooth varieties, arXiv preprint 0803.2105v1 (2008).
- [7] R. Crew, F -isocrystals and p -adic representations, in *Algebraic geometry (Brunswick, Maine, 1985), part 2*, Proc. Sympos. Pure. Math. 46, Amer. Math. Soc., Providence, 1987, 111–138.
- [8] A.J. de Jong, Smoothness, semi-stability and alterations, *Inst. Hautes Études Sci. Publ. Math.* **83** (1996), 51–93.
- [9] B. Green, M. Matignon, and F. Pop, On valued function fields, I, *Manuscripta Math.* **65** (1989), 357–376.
- [10] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math. 52, Springer-Verlag, New York, 1977.
- [11] N.M. Katz, Local-to-global extensions of representations of fundamental groups, *Ann. Inst. Fourier* **36** (1986), 69–106.
- [12] K.S. Kedlaya, A p -adic local monodromy theorem, *Ann. of Math.* **160** (2004), 93–184.
- [13] K.S. Kedlaya, Local monodromy for p -adic differential equations: an overview, *Intl. J. of Number Theory* **1** (2005), 109–154.
- [14] K.S. Kedlaya, Slope filtrations revisited, *Doc. Math.* **10** (2005), 447–525; errata, *ibid.* **12** (2007), 361–362.
- [15] K.S. Kedlaya, Swan conductors for p -adic differential modules, I: A local construction, *Alg. and Num. Theory* **1** (2007), 269–300.
- [16] K.S. Kedlaya, The p -adic local monodromy theorem for fake annuli, *Rend. Sem. Math. Padova* **118** (2007), 101–146.
- [17] K.S. Kedlaya, Semistable reduction for overconvergent F -isocrystals, I: Unipotence and logarithmic extensions, *Compos. Math.* **143** (2007), 1164–1212.
- [18] K.S. Kedlaya, Semistable reduction for overconvergent F -isocrystals, II: A valuation-theoretic approach, *Compos. Math.* **144** (2008), 657–672.
- [19] K.S. Kedlaya, Semistable reduction for overconvergent F -isocrystals, III: Local semistable reduction at monomial valuations, *Compos. Math.* **145** (2009), 143–172.
- [20] K.S. Kedlaya, *p -adic Differential Equations*, Cambridge Studies in Advanced Math. 125, Cambridge Univ. Press, Cambridge, 2010.

- [21] K.S. Kedlaya, Good formal structures for flat meromorphic connections, II: Excellent schemes, arXiv preprint 1001.0544v1 (2010).
- [22] K.S. Kedlaya, Swan conductors for p -adic differential modules, II: Global variation, arXiv preprint 0705.0031v3 (2008); to appear in *J. Inst. Math. Jussieu* (2011).
- [23] K.S. Kedlaya and L. Xiao, Differential modules on p -adic polyannuli, *J. Inst. Math. Jussieu* **9** (2010), 155–201; erratum, *ibid.* **9** (2010), 669–671.
- [24] G. Kempf, F.F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Math. 339, Springer-Verlag, Berlin, 1973.
- [25] W. Krull, Allgemeine Bewertungstheorie, *J. für Math.* **167** (1932), 160–196.
- [26] F.-V. Kuhlmann, Elimination of ramification I: The generalized stability theorem, *Trans. Amer. Math. Soc.* **362** (2010), 5697–5727.
- [27] Z. Mebkhout, Analogue p -adique du théorème de Turrittin et le théorème de la monodromie p -adique, *Invent. Math.* **148** (2002), 319–351.
- [28] J.S. Milne, *Étale Cohomology*, Princeton Math. Studies 33, Princeton Univ. Press, Princeton, 1980.
- [29] M. Raynaud and L. Gruson, Critères de platitude et de projectivité. Techniques de “platification” d’un module, *Invent. Math.* **13** (1971), 1–89.
- [30] P. Ribenboim, *The Theory of Classical Valuations*, Springer-Verlag, New York, 1999.
- [31] N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Math. 265, Berlin, 1972.
- [32] A. Shiho, Crystalline fundamental groups. II. Log convergent cohomology and rigid cohomology, *J. Math. Sci. Univ. Tokyo* **9** (2002), 1–163.
- [33] A. Shiho, On logarithmic extension of overconvergent isocrystals, arXiv preprint 0806.4394v2 (2009).
- [34] M. Temkin, Inseparable local uniformization, arXiv preprint 0804.1554v2 (2010).
- [35] N. Tsuzuki, On the Gysin isomorphism in rigid cohomology, *Hiroshima Math. J.* **29** (1999), 479–527.
- [36] O. Zariski, Local uniformization on algebraic varieties, *Annals of Math.* **41** (1940), 852–896.
- [37] O. Zariski and P. Samuel, *Commutative Algebra, Volume II*, reprint of the 1960 original, Graduate Texts in Math. 29, Springer-Verlag, New York, 1976.